Parabolic hyperbolic systems: lack of null-controllability in small time

Armand Koenig

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Université Côte d'Azur

Parabolic-hyperbolic systems

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Equation we are interested in:
$A, B \in \mathcal{M}_n(\mathbb{R})$, $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right)$ with $D + D^\top > 0$

$$\partial_t y(t,x) + A\partial_x y(t,x) - B\partial_{xx} y(t,x) = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{T}$$

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Question
Are these systems observable (equivalently: null-controllable) in $\omega \subset \mathbb{T}$?

$$|y(T,\cdot)|_{L^2(\mathbb{T})} \overset{?}{\leq} C|y|_{L^2([0,T]\times\omega)}$$

Fourier components, well-posedness

Fourier components
If $y(t, x) = \sum y_n(t) e^{inx}$

$$\partial_t y_n(t) + n^2 \left(B + \frac{i}{n} A \right) y_n(t) = 0$$

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Well-posedness
λ_{nk} eigenvalues of $B + \frac{i}{n} A$. Perturbation of B: $\lambda_{nk} \to \lambda_k \in \mathrm{Sp}(B)$

- If $\lambda_k > 0$: well-posed
- If $\lambda_k = 0$, $\lambda_{nk} \sim i\mu_k/n$: need $\mu_k \in \mathbb{R}$ (OK if A symmetric)

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Transport-like solutions
If $\lambda_{nk} \sim i\mu_k/n$, and y_{nk} is an associated eigenvector

$$y(t,x) = \sum_n a_n e^{inx - n^2 \lambda_{nk} t} y_{nk} \simeq \sum_n a_n e^{in(x - \mu_k t)} y_k$$

Not observable in small time.

Lack of small-time observability of the transport equation: Kafka's proof

Transport equation's lack of observability: Kafka's proof

- Equation $(\partial_t + \partial_x)y(t,x) = 0$, solutions: $y(t,x) = \sum_{n>0} a_n e^{in(x-t)}$
- Associated polynomial: $\tilde{y}(z) = \sum a_n z^n$ (imagine $z = e^{i(x-t)}$)

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- LHS of the observability inequality:

$$|y(T,\cdot)|^2_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \left|\sum a_n e^{in(x-T)}\right|^2 \mathrm{d}x = 2\pi \sum |a_n|^2 \geq C|\tilde{y}|^2_{L^2(D(0,1))}$$

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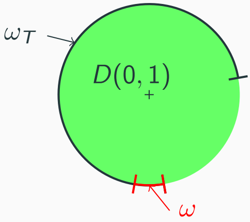
- RHS of the observability inequality:

$$|y|_{L^2([0,T]\times\omega)} \leq C \sup_{0<t<T} |y(t,\cdot)|_{L^\infty(\omega)} \leq C \sup_{0<t<T} |\tilde{y}|_{L^\infty(e^{-it}\omega)} \leq C|\tilde{y}|_{L^\infty(\omega_T)}$$



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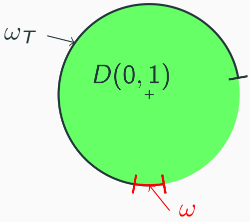
Conclusion
For every complex polynomial $\tilde{y}$:

$$|\tilde{y}|_{L^2(D(0,1))} \leq C|\tilde{y}|_{L^\infty(\omega_T)}$$

Does not hold if $\overline{\omega_T}$ is not the whole unit circle. $\square$



To the parabolic-hyperbolic systems

Parabolic hyperbolic systems as a perturbation of transport equation

All the answers in: Kato, **Perturbation Theory for Linear Operators**.

- Eigenvalues of $B + \frac{i}{n}A$: $\lambda_{nk} = i\mu_k/n + \rho_k(n)/n^2$ with $\rho_k(z) = \mathcal{O}(1)$
- (Generalized) eigenvectors: $y_{nk} = y_k(n)$ with $y_k(z) = y_k + o(1)$
- (Possible branch point at ∞)
- Particular solution:

$$y(t,x) = \sum a_n e^{in(x-\mu_k t)} \underbrace{e^{-t\rho_k(n)} y_k(n)}_{\text{error term}}$$

Managing the error terms

Theorem
Let $z \mapsto \gamma(z)$ *be (vector-valued) holomorphic and bounded for* $|z| > R$*. The Taylor series* $K_\gamma(z) = \sum \gamma(n) z^n$ *can be extended to a holomorphic function on* $\mathbb{C} \setminus [1, +\infty)$*.*

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Theorem
Let $z \mapsto \gamma(z)$ *be vector valued, holomorphic and bounded for* $|z| > R$. *Let* U *be a bounded open subset of* $\mathbb{C}$ *that is star-shaped with respect to* 0 *and* $K \subset\subset U$. *Then, for every polynomials* $\sum a_n z^n$*:*

$$\left|\sum \gamma(n) a_n z^n\right|_{L^\infty(K)} \leq C(K, V, \gamma) \left|\sum a_n z^n\right|_{L^\infty(U)}.$$

Proof.
Cauchy's integral formula + previous theorem. □

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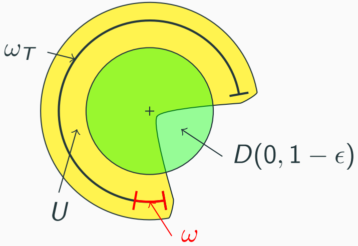
- RHS: previous theorem: $|y(t,\cdot)|_{L^\infty(\omega)} \leq C|\tilde{y}|_{L^\infty(U)}$
- LHS: error term does not decay too fast: $|y(T,\cdot)|_{L^2(\mathbb{T})} \geq C|\tilde{y}|_{L^2(D(0,1-\epsilon))}$



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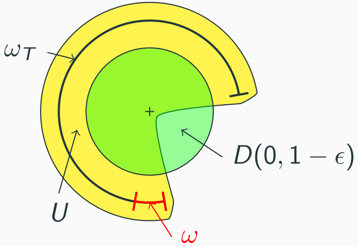

What we (don't) know

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- Controllable in large time ?
- Higher dimensions ?
- Non-constant coefficients ?

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That's all folks!