(Non) Null Controllability of the Fractional Heat Equation and of Related Equations

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The Problem of Controllability

Definition of the Controllability

Ω domain of $\mathbb{R}^n$, ω an open subset of Ω and $T > 0$.

Definition (Controllability of the heat equation on ω in time T)
For every initial condition $f_0 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = 1_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

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Theorem (Controllability of the heat equation (Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996))
Ω a C^2 bounded domain of $\mathbb{R}^n$, ω a non empty open subset of Ω, and $T > 0$. The heat equation is null-controllable on ω in time T.

Spectral Inequality

Theorem (Spectral inequality, Lebeau & Robbiano 1995)
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ϕ_k eigenfunction of $-\Delta$, eigenvalue λ_k.

$$\Big|\sum_{\lambda_k \le \mu} a_k \phi_k \Big|_{L^2(\Omega)} \le C e^{K\sqrt{\mu}} \Big|\sum_{\lambda_k \le \mu} a_k \phi_k \Big|_{L^2(\omega)}$$

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- Dissipation of the heat equation : $f_0 = \sum_{\lambda_k > \mu} a_k \phi_k$

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- Only depends on the spectral inequality
- E.g. also works for $\partial_t + (-\Delta)^\alpha$ (with $\alpha > 1/2$)

Examples of parabolic PDEs with little dissipation

Examples

- Fractional heat $(\partial_t + (-\Delta)^\alpha)f = 1_\omega u \quad (\alpha \leq 1/2)$
  Spectral inequality with $\sqrt{\mu}$, Dissipation with μ^α
- Grushin $(\partial_t - \partial_x^2 - x^2\partial_y^2)f = 1_\omega u$
  Spectral inequality with μ, Dissipation with μ
- Kolmogorov $(\partial_t - \partial_v^2 - v^2\partial_x)f = 1_\omega u$
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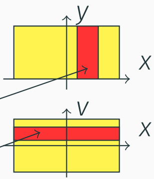

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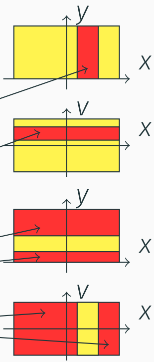

Fractional Heat Equation ($\alpha < 1/2$)

$\Omega = \mathbb{R}$, $\omega = \{|x| > \epsilon\}$, $\Re(z) > 0$.

Non-null-controllability of $\partial_t + z(-\Delta)^\alpha$

- Controllability $\Leftrightarrow$ observability:
  $(\partial_t + \bar{z}(-\Delta)^\alpha)g = 0 \implies |g(T,\cdot)|_{L^2(\Omega)} \leq C|g|_{L^2([0,T]\times\omega)}$

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- Saddle point method:

$$g(t,x) = \mathcal{O}\left(\frac{1}{|x|^\infty} e^{-ct/h}\right) \qquad |x| > \epsilon$$

$$g(t,x) = e^{ix\xi_0/h - x^2/2h - \mathcal{O}(h^{-2\alpha})} \qquad |x| < \frac{\xi_0}{4}$$

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Observability inequality applied with $f(t,y) = \sum a_n e^{-nt} e^{iny}$:

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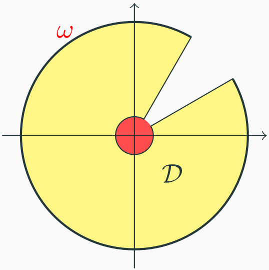
Let $z = e^{-t+iy}$ and $f(z) = \sum a_n z^{n-1}$

$$\begin{aligned} &\int_{D(0,e^{-T})} |f(z)|^2 \,\mathrm{d}\lambda(z) \\ &\leq 2\pi \sum |a_n|^2 e^{-2nT} \\ &\leq 2\pi C \int_{\mathcal{D}} |f(z)|^2 \,\mathrm{d}\lambda(z) \end{aligned}$$



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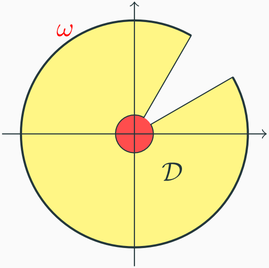

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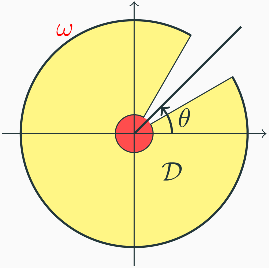

Untrue thanks to Runge's theorem (take $f_k \longrightarrow 1/z$ uniformly on every compact of $\mathbf{C} \setminus e^{i\theta}\mathbf{R}_+$) $\square$

Degenerate Parabolic Equations

Kolmogorov & Grushin

- Link Grushin/half-heat by looking at special solutions

$$f(t,x,y) = \sum a_n e^{-|n|t} e^{-|n|x^2/2+iny}$$

- (Same idea for Kolmogorov/rotated fractional heat)

Kolmogorov & Grushin

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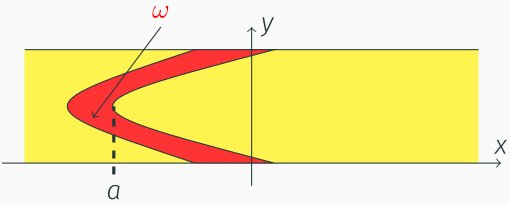
- (Same idea for Kolmogorov/rotated fractional heat)
- For Grushin: lack of small time null-controllability if $\{x = 0\} \not\subset \bar{\omega}$
- Plus null-controllability as a consequence of a result by Beauchard $\implies$ accurate minimal time of null-controllability for $\omega = \{f_1(y) < x < f_2(y)\}$. With $a = \max(\sup f_2^-, \sup f_1^+)$, $T_{\min} = a^2/2$.



Conclusion

Some open problems

- Results for half-heat limited to $\Omega = \mathbb{T}$ (no $\mathbb{R}$)
- Results for Grushin and Kolmogorov limited to the potential x^2

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That's all folks!