Control of the fractional heat equation and related equations

Armand Koenig

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Spectral Theory and Geometry

Introduction

Control of the heat equation

Ω domain of $\mathbb{R}^n$, ω open subset of Ω and $T > 0$.

Definition (Null-controllability of the heat equation on ω in time T)

For every initial condition $f_0 \in L^2(\Omega)$, there exists a control $u \in L^2([0,T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = 1_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

satisfies $f(T, \cdot) = 0$ on Ω.

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Theorem (Control of the heat equation (Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996))

Ω a C^2 bounded connected open subset of $\mathbb{R}^n$, ω a non-empty open subset of Ω, and $T > 0$. The heat equation is null-controllable ω in time T.

Observability: a notion dual to controllability

Theorem

- *The equation* $\partial_t f - \Delta f = 1_\omega u$ *is null-controllable on* ω *in time* T *if and only if*
- *for every solution of* $\partial_t g - \Delta g = 0$,

$$|g(T,\cdot)|^2_{L^2(\Omega)} \leq C|g|^2_{L^2([0,T]\times\omega)}.$$

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Remark

Duality observability/controlability: happens for every linear equation.
$(\partial_t + A)f = Bu$ is null-controllable in time T if and only if for every g_0,

$$|e^{-TA^*}g_0|^2 \leq C\int_0^T |B^*e^{-tA^*}g_0|^2 \,\mathrm{d}t$$

Lebeau and Robbiano's method

Theorem (Spectral inequality, Jerison-Lebeau 1996)

Ω a C^2 connected bounded open subset of $\mathbb{R}^n$, ω a non-empty open subset of Ω.

ϕ_k the eigenfunctions of $-\Delta$, with eigenvalues λ_k.

$$\Big|\sum_{\lambda_k \leq \mu} a_k\phi_k\Big|_{L^2(\Omega)} \leq Ce^{K\sqrt{\mu}}\Big|\sum_{\lambda_k \leq \mu} a_k\phi_k\Big|_{L^2(\omega)}$$

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- Allows to steer to zero the frequencies $\lambda_k \leq \mu$
- Dissipation of the heat equation: $f_0 = \sum_{\lambda_k > \mu} a_k \phi_k$

$$|e^{t\Delta} f_0|^2_{L^2(\Omega)} \leq e^{-2\mu t} |f_0|^2_{L^2(\Omega)}$$

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- Depends only on the spectral inequality

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- Dissipation $\gg$ spectral inequality $\implies$ null-controllability
- Depends only on the spectral inequality
- Also proves the null-controllability $\partial_t + (-\Delta)^\alpha$ with $\alpha > 1/2$
- What happens if $\alpha \leq 1/2$?

Equations with low dissipation

Fractional heat $(\partial_t + (-\Delta)^\alpha)f = 1_\omega u \qquad (\alpha \leq 1/2)$

- Spectral inequality: $e^{K\sqrt{\mu}}$, dissipation: $e^{-\mu^\alpha}$
- Not null-controllable [Micu-Zuazua, Miller, K]

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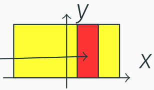
Baouendi-Grushin heat $(\partial_t - \partial_x^2 - x^2\partial_y^2) f = 1_\omega u$

- Spectral inequality: $e^{K\mu}$, dissipation: $e^{-\mu}$
- Null-controllable only in large enough time if ω



  [Beauchard-Cannarsa-Guglielmi, Beauchard-Miller-Morancey, Beauchard-Dardé-Ervedoza]
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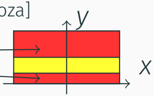

  [K, Duprez-K]

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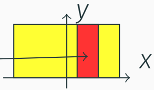

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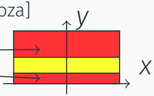

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Kolmogorov-type $(\partial_t - \partial_v^2 + v^2\partial_x)f = 1_\omega u$

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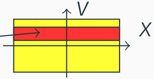

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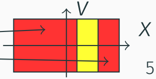

  [K]

Plan

Fractional heat equation and Kolmogorov-type equation

Half-heat equation and Baouendi-Grushin heat equation

Conclusion

Fractional heat equation and Kolmogorov-type equation

Fractional heat equation

- Fractional Laplacian: $(-\Delta)^{\alpha} f = \mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}f(\xi))$
- Control system: $(\partial_t + (-\Delta)^{\alpha})f(t,x) = 1_{\omega} u, \quad x \in \mathbb{R}$

Fractional heat equation

Generalized Fractional heat equation

- Fractional Laplacian: $\rho(\sqrt{-\Delta}) = \mathcal{F}^{-1}(\rho(|\xi|)\mathcal{F}f(\xi))$
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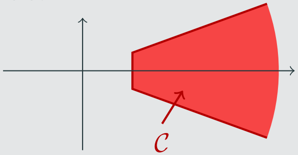
Theorem (Non-null-controllability of the fractional heat equation (K 2019))

Let $K > 0$ and $\mathcal{C} = \{\Re(\xi) > K, |\Im(\xi)| < K\Re(\xi)\}$. Let $\rho\colon \mathcal{C} \cup \mathbb{R}_+ \to \mathbb{C}$ such that

- $\inf_{\xi > 0} \Re(\rho(\xi)) > -\infty$
- *ρ is holomorphic on $\mathcal{C}$*
- *$\rho = o(|\xi|)$ for $|\xi| \to +\infty$, $\xi \in \mathcal{C}$*



Let $T > 0$ and ω a strict open subet of $\mathbb{R}$. The equation

$$(\partial_t + \rho(\sqrt{-\Delta}))f = 1_\omega u$$

is not null-controllable on ω in time T.

Fractional heat equation: non-null-controllability

$\Omega = \mathbb{R}$, $\omega = \{|x| > \epsilon\}$.

Non-null-controllability of $\partial_t + \rho(\sqrt{-\Delta})$.

- Controlability $\Leftrightarrow$ observability:
  $(\partial_t + \bar{\rho}(\sqrt{-\Delta}))g = 0 \implies |g(T, \cdot)|_{L^2(\Omega)} \leq C|g|_{L^2([0,T]\times\omega)}$

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$$g(t,x) = c_h e^{ix\xi_0/h - x^2/2h} \int_{\mathbb{R}} \chi(\xi) e^{-(\xi - ix)^2/2h - t\bar{\rho}((\xi+\xi_0)/h)} \mathrm{d}\xi$$

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- Saddle point method:

$$g(t,x) = \mathcal{O}\left(\frac{1}{|x|^\infty} e^{-ct/h}\right) \qquad |x| > \epsilon$$

$$g(t,x) = e^{ix\xi_0/h - x^2/2h - \text{“}O(\rho(1/h))\text{”}} \qquad |x| < \delta$$

□

Kolmogorov-type equation

A Kolmogorov-type equation

$$(\partial_t - \partial_v^2 + v^2\partial_x)f(t,x,v) = 1_{\omega}u(t,x,v),\ x \in \mathbb{R}, v \in \mathbb{R}$$

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«Embedding» of the fractional heat in the Kolmogorov-type equation

- For $\xi \in \mathbb{R}$, $e^{-\sqrt{i\xi}v^2/2+ix\xi}$ eigenfunction, eigenvalue $\sqrt{i\xi}$
- Particular solution: $g(t,x,v) = \int_{\mathbb{R}} a(\xi)e^{ix\xi-\sqrt{i\xi}(t+v^2/2)}\,\mathrm{d}\xi$
- In x-variable: solution of $(\partial_t + \sqrt{i}(-\Delta_x)^{1/4})g(t,x) = 0$

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Theorem (Kolmogorov-type controlled on vertical strip)

Let $T > 0$, ω_v a strict open subset $\mathbb{R}$ and $\omega = \omega_v \times \mathbb{R}$. The Kolmogorov-type equation is not null-controllable on ω in time T.

Kolmogorov-type equation

A Kolmogorov-type equation

$$(\partial_t - \partial_v^2 + v^1\partial_x) f(t,x,v) = 1_\omega u(t,x,v),\ x\in\mathbb{R}, v\in\mathbb{R}_+$$

«Embedding» of the fractional heat in the Kolmogorov-type equation

- For $\xi\in\mathbb{R}$, $\mathrm{Ai}(\xi^{1/3}e^{-i\pi/6}v-\mu_0)$ eigenfunction, eigenvalue $e^{i\pi/3}\mu_0\xi^{2/3}$
- Particular solution: $g(t,x,v) = \int_{\mathbb{R}} a(\xi)e^{ix\xi - te^{i\pi/3}\mu_0\xi^{2/3}} Ai(e^{-i\pi/6}\xi^{1/3}v-\mu_0)\,\mathrm{d}\xi$
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More equations

- 1-D torus in x, segment in v
- $(\partial_t^2 - \partial_t^2\partial_x^2 - \partial_x^2)f(t,x) = 1_\omega u(t,x)$, perturbation of $(-\Delta)^0$
- ...?

Half-heat equation and Baouendi-Grushin heat equation

Half-heat equation

- Half-Laplacian: $\sqrt{-\Delta}\left(\sum_{n\in\mathbb{Z}}\widehat{f}(n)e^{inx}\right) = \sum_{n\in\mathbb{Z}}|n|\widehat{f}(n)e^{inx}$
- Control system: $(\partial_t + \sqrt{-\Delta})f(t,x) = 1_{\omega}u, \quad x \in \mathbb{T}$

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Theorem (Non-null-controllability)

Let $T > 0$ and ω a strict open subset of $\mathbb{T}$. The half-heat equation

$$(\partial_t + \sqrt{-\Delta})f = 1_\omega u$$

is not null-controllable on ω in time T.

Non-null-controllability of the half heat

Proof.
Test observability inequality with $g(t,x) = \sum_{n>0} a_n e^{-nt} e^{inx}$:

$$\sum_{n>0} |a_n|^2 e^{-2nT} \leq C \int_{[0,T]\times\omega} \Big|\sum_{n>0} a_n e^{-nt} e^{inx}\Big|^2 \,\mathrm{d}t\,\mathrm{d}x$$

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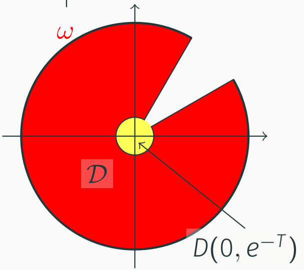
- Change of variables: $z = e^{-t+ix}$

$$|g|^2_{L^2([0,T]\times\omega)} = \int_{\mathcal{D}} \left| \sum_{n>0} a_n z^{n-1} \right|^2 \mathrm{d}\lambda(z)$$

- Computation in polar coordinates:

$$|g(T,\cdot)|^2_{L^2(\mathbb{T})} \geq \pi^{-1} \int_{D(0,e^{-T})} \left| \sum_{n>0} a_n z^{n-1} \right|^2 \mathrm{d}\lambda(z)$$



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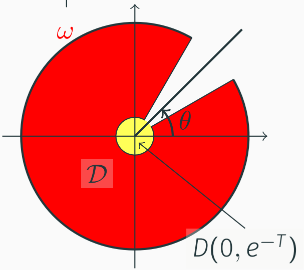

- Observability $\Rightarrow$ for every $p \in \mathbb{C}[X]$, $|p|_{L^2(D(0,e^{-T}))} \leq C|p|_{L^2(\mathcal{D})}$
- Not true according to the Runge theorem (there exists $p_k(z) \longrightarrow 1/z$ away from $\mathbb{C} \setminus e^{i\theta}\mathbb{R}_+$) $\square$

Équation de Baouendi-Grushin

$$(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 1_\omega u(t,x,y),\ x \in \mathbb{R}, y \in \mathbb{T}$$

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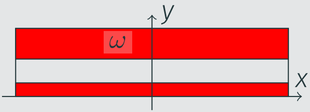
«Embedding» of the half-heat in the Baouendi-Grushin heat equation

- For $n \in \mathbb{N}$, $e^{-nx^2/2+iny}$ eigenfunction, eigenvalue n
- Particular solutions: $g(t,x,y) = \sum_{n>0} a_n e^{-nt-nx^2/2+iny}$
- In the y-variable: solution of the half-heat equation

Control of the Baouendi-Grushin heat equation

Theorem (Baouendi-Grushin heat equation on horizontal strip)



$\omega = \mathbb{R} \times \omega_y$

Not null-controllable on ω (whatever $T > 0$)

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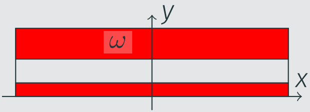

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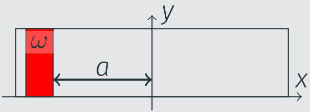

$\omega = (a, b) \times \mathbb{R}$

Null-controllable on ω *if and only if* $T > a^2/2$

Control of the Baouendi-Grushin heat equation

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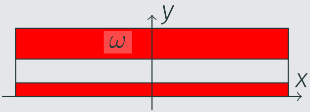

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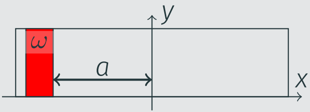
Not null-controllable on ω *(whatever* $T > 0$*)*

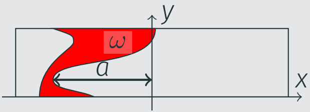
Theorem (Beauchard-Dardé-Ervedoza 2018)



$\omega = (a, b) \times \mathbb{R}$

Null-controllable on ω *if and only if* $T > a^2/2$

Theorem (Duprez-K 2018)



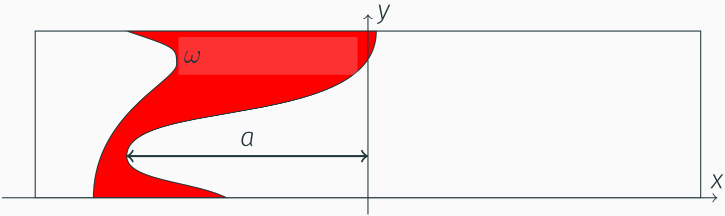
$\omega = \{\gamma_1(y) < x < \gamma_2(y)\}$, $a = \max(\sup(\gamma_2^-), \sup(\gamma_1^+))$

Null-controllable on ω *if* $T > a^2/2$

Not null-controllable on ω *if* $T < a^2/2$

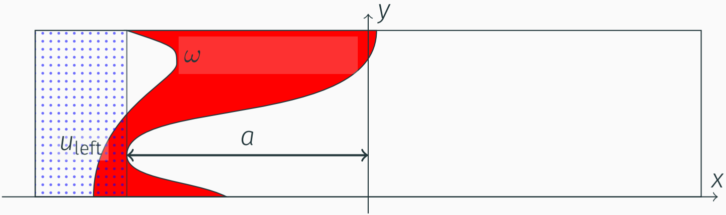
Large time null-controllabilty

Proof.



- Null-controlability in large time known on vertical strip

Proof.



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- u_{left} control on a vertical strip on the left (possible if $T > a^2/2$)

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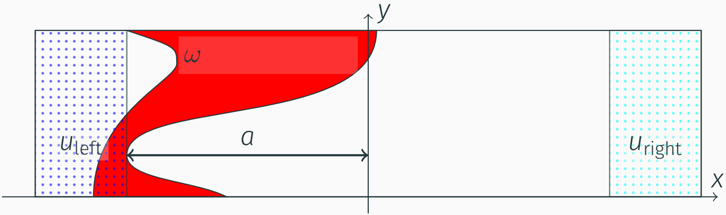

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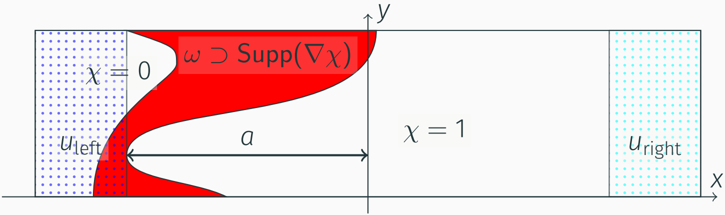

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- u_{left} control on a vertical strip on the left (possible if $T > a^2/2$)
- u_{right} control on a vertical strip on the right (possible if $T > a^2/2$)
- χ cutoff with $\text{Supp}(\nabla\chi) \subset \omega$, $\chi = 0$ «left of ω» and $\chi = 1$ «right of ω»
- $f := \chi f_{\text{left}} + (1-\chi) f_{\text{right}}$.
  $(\partial_t - \partial_x^2 - x^2\partial_y^2) f = \chi u_{\text{left}} + (1-\chi) u_{\text{right}} +$ terms involving $\nabla\chi, \Delta\chi$ □

Non-null-controllability in small time

Proof.

- Particular solutions: $g(t,x,y) = \sum_{n>0} a_n e^{-nt-nx^2/2+iny}, \quad p(z) = \sum_{n>0} a_n z^{n-1}$
- Lower bound LHS: $|g(T,\cdot,\cdot)|_{L^2}^2 \geq \sum \frac{|a_n|^2}{\sqrt{n}} e^{-2nT} \geq c|p|^2_{L^2(D(0,e^{-T}))}$

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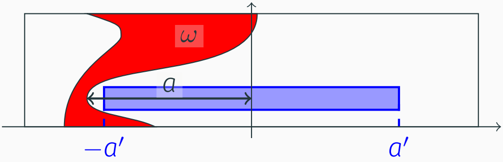

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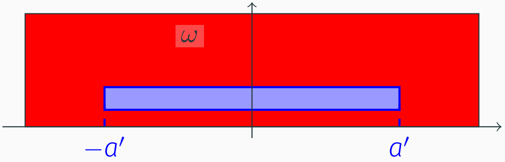

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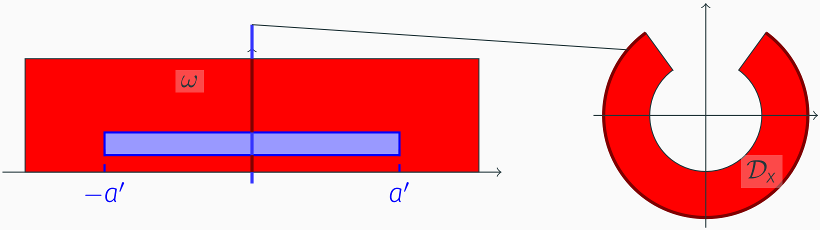

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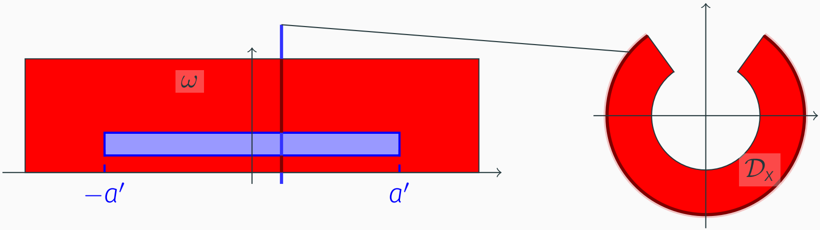

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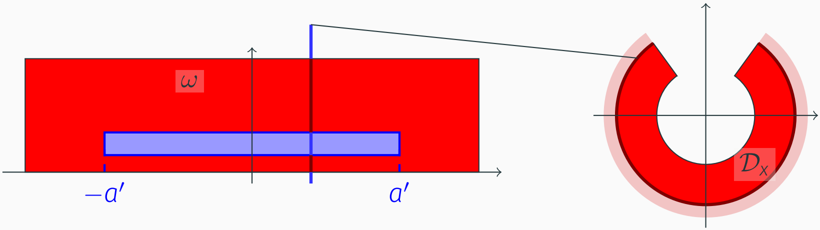

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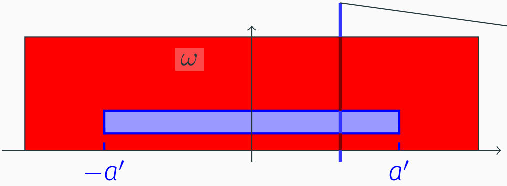

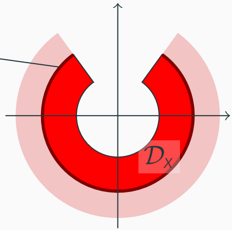

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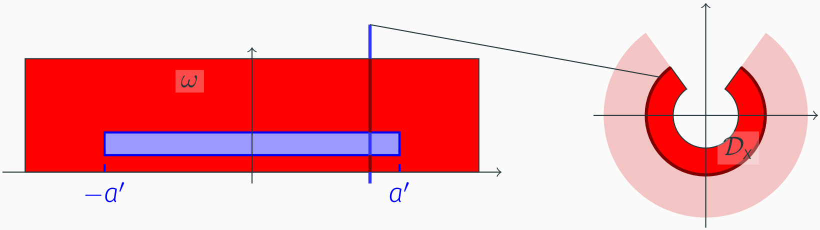

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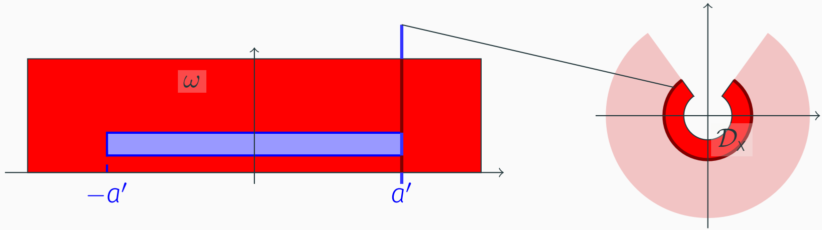

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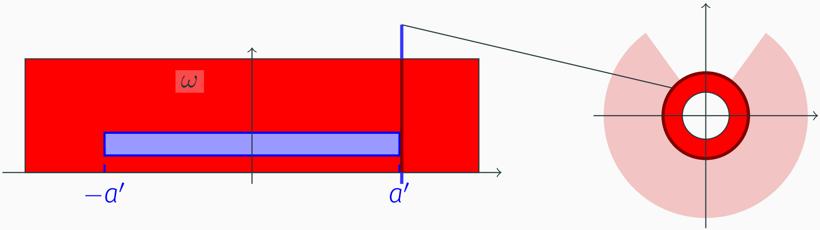

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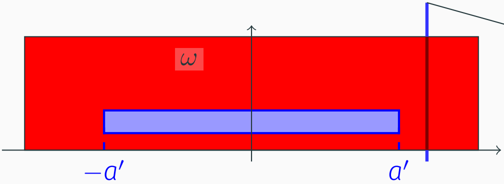

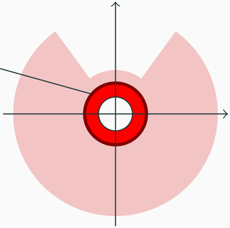

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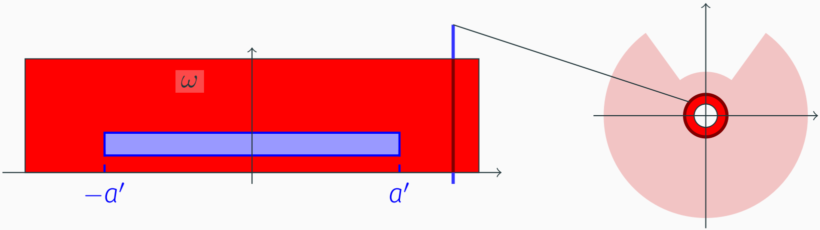

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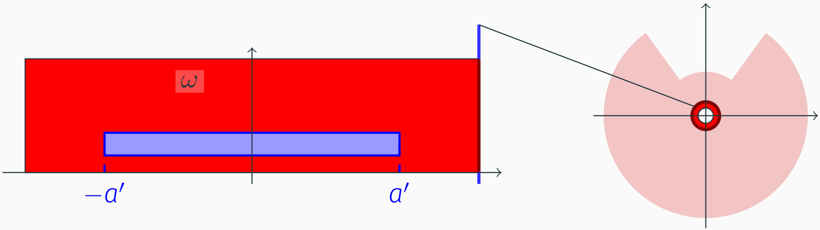

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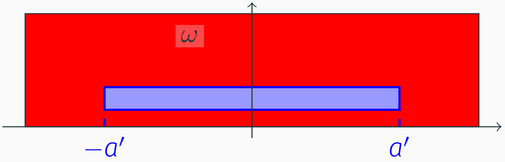
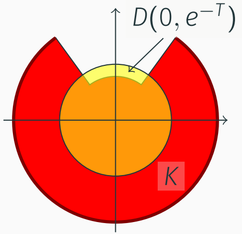
- Observability $\Rightarrow$ for every $p\in\mathbb{C}[X]$, $|p|_{L^2(D(0,e^{-T}))} \leq C|p|_{L^\infty(K)}$ $\square$





Error terms

Baouendi-Grushin heat on a bounded domain

- $(\partial_t - \partial_x^2 - x^2\partial_y^2)g(t,x,y) = 0,\ x \in]-1,1[,\ y \in \mathbb{T}$, Dirichlet boundary conditions
- Eigenfunction: $v_n(x) = w_n(x)e^{-nx^2/2+iny}$, eigenvalue: $\lambda_n = n+\rho_n$
- Particular solutions: $g(t,x,y) = \sum_{n>0} a_n e^{-nx^2/2-nt+iny} w_n(x)e^{-\rho_n t}$

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Definition

$(\gamma(n))$ a sequence. H_γ the operator on polynomials

$$H_\gamma \colon \sum a_n z^n \mapsto \sum \gamma(n) a_n z^n$$

Find continuity-like estimates for H_γ in the right norms

Estimations on the operators H_γ

Theorem

γ holomorphic bounded on $\{\Re(z) > C\}$. K a compact subset of $\mathbb{C}$. $U \supset K$, open, star-shaped with respect to 0. $p = \sum a_n z^n \in \mathbb{C}[X]$

$$|H_\gamma p|_{L^\infty(K)} \leq C|p|_{L^\infty(U)} \qquad \left|\sum_{n>0} \gamma(n) a_n z^n\right|_{L^\infty(K)} \leq C\left|\sum_{n>0} a_n z^n\right|_{L^\infty(U)}$$

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- With $K_\gamma(\zeta) = \sum \gamma(n)\zeta^n, \quad H_\gamma p(z) = \dfrac{1}{2i\pi} \displaystyle\oint_{\partial D} \frac{1}{\zeta} K_\gamma\left(\frac{z}{\zeta}\right) p(\zeta)\,\mathrm{d}\zeta$
- Theorem : $K_\gamma(\zeta)$ extends holomorphically to $\zeta \notin [1, +\infty[$
- Change of integration path:

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□

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Apply this to $\gamma(n) = w_n(x)e^{-\rho_n t}$:

$$\int_{\mathcal{D}_x} \left|\sum_{n>0} a_n e^{-nx^2/2 - nt + iny} w_n(x) e^{-\rho_n t}\right|^2 \mathrm{d}t\,\mathrm{d}y \leq C\,\mathrm{area}(\mathcal{D}_x)\left|\sum a_n z^{n-1}\right|^2_{L^\infty(U)}$$

Spectral analysis of the harmonic oscillator on $(-1, 1)$

Semiclassical non self-adjoint harmonic oscillator

$$\Re(z) > 0, \qquad \mathcal{P}_h := -\partial_x^2 + z^2 x^2, \quad D(\mathcal{P}_h) = H^2(-1,1) \cap H_0^1(-1,1)$$

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Let λ_h be the (holomorphic continuation of the) first eigenvalue of $\mathcal{P}_h$. Let $\theta \in (0, \pi/2)$. Then for $|h| \to 0$, $|\arg(h)| < \theta$, $\lambda_h \sim h + e^{-1/h}\left(4\sqrt{\frac{h}{\pi}} + \dots\right)$.

Spectral analysis of the harmonic oscillator on $(-1,1)$

Semiclassical non self-adjoint harmonic oscillator

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Sketch of the proof by ODE techniques.

- For $\lambda \in \mathbb{C}$, write solution of $-h^2u'' + x^2u = \overbrace{\lambda}^{h(1+2\rho)} u$:
$$u_\pm(x) = e^{-x^2/2h}\int_{\text{some complex path } \Gamma_\pm} e^{-(t^2/4+xt)/h-(1+\rho)\ln(t)}\,\mathrm{d}t$$
- λ eigenvalue "$\Leftrightarrow$" $\Phi(h,\rho) := (1+e^{i\pi\rho})u_+(-1) - (1+e^{-i\pi\rho})u_-(-1) = 0$
- Solve the previous implicit equation (for $\rho = \rho(h)$ with a Newton scheme: $\rho_0(h) = 0$, $\rho_{n+1}(h) = \rho_n(h) - \partial_\rho\Phi(h,\rho_n(h))^{-1}\Phi(h,\rho_n(h))$
- Saddle point method: estimate for Newton and $\rho_1(h) \sim e^{-1/h}2(\pi h)^{-1/2}$ $\square$

Conclusion

Control of parabolic equation with low dissipation

Low diffusion $\implies$ not null-controllable in arbitrarily small time

- Fractional heat equation with low dissipation: not null-controllable
- Baouendi-Grushin heat: geometric condition for null-controllability
  Relevant quantity: maximum Agmon distance between $\{x = 0\}$ and ω?
- Kolmogorov-type: geometric control condition for nul-controllability?

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Open problem

- Fractional heat on manifolds? Lower order terms?
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That’s all folks!