Control of parabolic PDEs

Phd student's Seminar

Armand Koenig

January 9, 2020

Introduction

Context & problem

Control of PDEs

Ω domain of $\mathbb{R}^n$, ω open subset of Ω and $T > 0$.

Definition (Null-controllability of the heat equation on ω in time T)

For every initial condition $f_0 \in L^2(\Omega)$, there exists a function $u \in L^2([0,T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = 1_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

satisfies $f(T, \cdot) = 0$ on Ω.

Ω domain of $\mathbb{R}^n$, ω open subset of Ω and $T > 0$.

Definition (Null-controllability of the heat equation on ω in time T)

For every initial condition $f_0 \in L^2(\Omega)$, there exists a function $u \in L^2([0, T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = 1_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

satisfies $f(T, \cdot) = 0$ on Ω.

Theorem (Null-controllability of the heat equation (Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996))

Ω a connected open bounded subset of $\mathbb{R}^n$ of class C^2, ω a non-empty open subset of Ω, et $T > 0$. The heat equation is null-controllable on ω in time T.

Theorem (Observability $\Leftrightarrow$ Controllability)

- *The equation $\partial_t f - \Delta f = 1_\omega u$ is null-controllable in time T*

  if and only if

- *for every solution of $\partial_t g - \Delta g = 0$,*

$$|g(T,\cdot)|^2_{L^2(\Omega)} \leq C|g|^2_{L^2([0,T]\times\omega)}.$$

Observability: a dual notion to the controllability

Theorem (Observability $\Leftrightarrow$ Controllability)

- *The equation* $\partial_t f - \Delta f = 1_\omega u$ *is null-controllable in time* T

  if and only if

- *for every solution of* $\partial_t g - \Delta g = 0$,

$$|g(T,\cdot)|^2_{L^2(\Omega)} \leq C|g|^2_{L^2([0,T]\times\omega)}.$$

Proof.
Integration by parts + Riesz representation theorem in Hilbert spaces

Alternatively: $\mathrm{Range}(\Phi_2) \subset \mathrm{Range}(\Phi_3) \Leftrightarrow |\Phi_2^* x| \leq C|\Phi_3^* x|$ $\square$

Remark
Duality observability/controllability: general phenomenon

Lebeau-Robbiano Method

Theorem (Spectral inequality, Lebeau & Robbiano 1995)

Ω connected C^2 open bounded subset of $\mathbb{R}^n$, ω a non-empty open subset of Ω. ϕ_k eigenfunctions of $-\Delta$, of eigenvalues λ_k.

$$\Big|\sum_{\lambda_k \leq \mu} a_k \phi_k\Big|_{L^2(\Omega)} \leq C e^{K\sqrt{\mu}} \Big|\sum_{\lambda_k \leq \mu} a_k \phi_k\Big|_{L^2(\omega)}$$

Lebeau-Robbiano Method

Theorem (Spectral inequality, Lebeau & Robbiano 1995)

Ω connected C^2 open bounded subset of $\mathbb{R}^n$, ω a non-empty open subset of Ω. ϕ_k eigenfunctions of $-\Delta$, of eigenvalues λ_k.

$$\Big| \sum_{\lambda_k \leq \mu} a_k \phi_k \Big|_{L^2(\Omega)} \leq C e^{K\sqrt{\mu}} \Big| \sum_{\lambda_k \leq \mu} a_k \phi_k \Big|_{L^2(\omega)}$$

- Allows to kills frequencies $\lambda_k \leq \mu$ to 0
- Dissipation of the heat equation: $f_0 = \sum_{\lambda_k > \mu} a_k \phi_k$

$$|e^{t\Delta} f_0|^2_{L^2(\Omega)} \leq e^{-2\mu t} |f_0|^2_{L^2(\Omega)}$$

Lebeau-Robbiano Method

Theorem (Spectral inequality, Lebeau & Robbiano 1995)

Ω connected C^2 open bounded subset of $\mathbb{R}^n$, ω a non-empty open subset of Ω. ϕ_k eigenfunctions of $-\Delta$, of eigenvalues λ_k.

$$\Big|\sum_{\lambda_k \leq \mu} a_k\phi_k\Big|_{L^2(\Omega)} \leq Ce^{K\sqrt{\mu}}\Big|\sum_{\lambda_k \leq \mu} a_k\phi_k\Big|_{L^2(\omega)}$$

- Allows to kills frequencies $\lambda_k \leq \mu$ to 0
- Dissipation of the heat equation: $f_0 = \sum_{\lambda_k > \mu} a_k\phi_k$

$$|e^{t\Delta}f_0|^2_{L^2(\Omega)} \leq e^{-2\mu t}|f_0|^2_{L^2(\Omega)}$$

- Dissipation $\gg$ spectral inequality $\implies$ null-controllability
- Only depends on the spectral inequality

Lebeau-Robbiano Method

Theorem (Spectral inequality, Lebeau & Robbiano 1995)

Ω connected C^2 open bounded subset of $\mathbb{R}^n$, ω a non-empty open subset of Ω. ϕ_k eigenfunctions of $-\Delta$, of eigenvalues λ_k.

$$\Big|\sum_{\lambda_k\leq\mu} a_k\phi_k\Big|_{L^2(\Omega)} \leq Ce^{K\sqrt{\mu}}\Big|\sum_{\lambda_k\leq\mu} a_k\phi_k\Big|_{L^2(\omega)}$$

- Allows to kills frequencies $\lambda_k \leq \mu$ to 0
- Dissipation of the heat equation: $f_0 = \sum_{\lambda_k>\mu} a_k\phi_k$

$$|e^{t\Delta}f_0|^2_{L^2(\Omega)} \leq e^{-2\mu t}|f_0|^2_{L^2(\Omega)}$$

- Dissipation $\gg$ spectral inequality $\implies$ null-controllability
- Only depends on the spectral inequality
- Also proves null-controllability of $\partial_t + (-\Delta)^\alpha$ if $\alpha > 1/2$
- Equation with low diffusion: dissipation $\lesssim$ spectral inequality

Examples of equations with low diffusion

Fractional heat $(\partial_t + (-\Delta)^\alpha)f = 1_\omega u \qquad (\alpha \leq 1/2)$

- Spectral inequality in $\sqrt{\mu}$, dissipation in μ^α
- Not null-controllable [Micu-Zuazua, Miller]

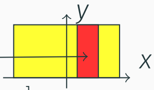
Examples of equations with low diffusion

Fractional heat $(\partial_t + (-\Delta)^\alpha)f = 1_\omega u \qquad (\alpha \leq 1/2)$

- Spectral inequality in $\sqrt{\mu}$, dissipation in μ^α
- Not null-controllable [Micu-Zuazua, Miller]

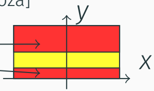
Grushin $(\partial_t - \partial_x^2 - x^2\partial_y^2)f = 1_\omega u$

- Spectral inequality in μ, dissipation in μ
- Null-controllable only in large enough time if ω



[Beauchard-Cannarsa-Guglielmi, Beauchard-Miller-Morancey, Beauchard-Dardé-Ervedoza]

- Never null-controllable if ω



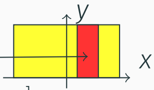
Examples of equations with low diffusion

Fractional heat $(\partial_t + (-\Delta)^\alpha)f = 1_\omega u \qquad (\alpha \leq 1/2)$

- Spectral inequality in $\sqrt{\mu}$, dissipation in μ^α
- Not null-controllable [Micu-Zuazua, Miller]

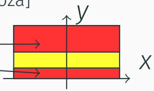
Grushin $(\partial_t - \partial_x^2 - x^2\partial_y^2)f = 1_\omega u$

- Spectral inequality in μ, dissipation in μ
- Null-controllable only in large enough time if ω



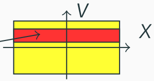
[Beauchard-Cannarsa-Guglielmi, Beauchard-Miller-Morancey, Beauchard-Dardé-Ervedoza]

- Never null-controllable if ω



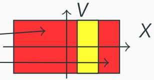
Kolmogorov $(\partial_t - \partial_v^2 + v^2\partial_x)f = 1_\omega u$

- Spectral inequality in μ, dissipation in $\sqrt{\mu}$
- Null-controllable only in large enough time if ω



[Beauchard-Zuazua, Beauchard, Beauchard-Helffer-Henry-Robbiano]

- Never null-controllable if ω



Possible obstructions to the null-controllability

Concentration of eigenfunctions

Example: Grushin equation

$$(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 1_{\omega}u(t,x,y),\ x \in \mathbb{R}, y \in \mathbb{T}$$

Example: Grushin equation

$$(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 1_\omega u(t,x,y),\ x \in \mathbb{R}, y \in \mathbb{T}$$

Concentration of eigenfunctions

- For $n \in \mathbb{N}$, $e^{-nx^2/2+iny}$ eigenfunction, with eigenvalue n

Concentration of eigenfunctions

Example: Grushin equation

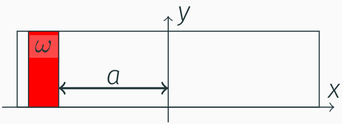
$$(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 1_\omega u(t,x,y),\ x \in \mathbb{R}, y \in \mathbb{T}$$

Concentration of eigenfunctions

- For $n \in \mathbb{N}$, $e^{-nx^2/2+iny}$ eigenfunction, with eigenvalue n

  $\omega = (a,b) \times \mathbb{T}$
- $|e^{-nT-nx^2/2+iny}|_{L^2(\mathbb{R}\times\mathbb{T})} = cn^{-1/4}e^{-nT}$

  $|e^{-nt-nx^2/2+iny}|_{L^2([0,T]\times\omega)} \approx cn^{-1/2}e^{-na^2/2}$



Concentration of eigenfunctions

Example: Grushin equation

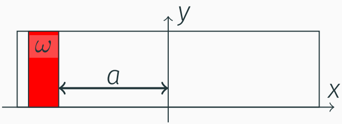
$$(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 1_{\omega}u(t,x,y),\ x \in \mathbb{R}, y \in \mathbb{T}$$

Concentration of eigenfunctions

- For $n \in \mathbb{N}$, $e^{-nx^2/2+iny}$ eigenfunction, with eigenvalue n

  $\omega = (a,b) \times \mathbb{T}$

- $|e^{-nT-nx^2/2+iny}|_{L^2(\mathbb{R}\times\mathbb{T})} = cn^{-1/4}e^{-nT}$
  $|e^{-nt-nx^2/2+iny}|_{L^2([0,T]\times\omega)} \approx cn^{-1/2}e^{-na^2/2}$



- Observability inequality untrue if $T < a^2/2$
- We can prove null-controllability if $T > a^2/2$ (much harder)
- Surprising: minimal time for null-controllability

Possible obstructions to the null-controllability

Weak Diffusion

Half-heat equation

- Half-laplace operator: $\sqrt{-\Delta}\left(\sum_{n\in\mathbb{Z}}\widehat{f}(n)e^{inx}\right) = \sum_{n\in\mathbb{Z}}|n|\widehat{f}(n)e^{inx}$
- Control system: $(\partial_t + \sqrt{-\Delta})f(t,x) = 1_{\omega}u, \quad x \in \mathbb{T}$

Half-heat equation

- Half-laplace operator: $\sqrt{-\Delta}\left(\sum_{n\in\mathbb{Z}}\widehat{f}(n)e^{inx}\right) = \sum_{n\in\mathbb{Z}}|n|\widehat{f}(n)e^{inx}$
- Control system: $(\partial_t + \sqrt{-\Delta})f(t,x) = 1_\omega u, \quad x \in \mathbb{T}$

Theorem (Lack of null-controllability)

Let $T > 0$ and ω a strict open subset of $\mathbb{T}$. The half-heat equation

$$(\partial_t + \sqrt{-\Delta})f = 1_\omega u$$

is not null-controllable on ω in time T.

Lack of null-controllability of half-heat

Proof.
Test observability inequality against $g(t,x) = \sum_{n>0} a_n e^{-nt} e^{inx}$:

$$\sum_{n>0} |a_n|^2 e^{-2nT} \leq C \int_{[0,T]\times\omega} \left| \sum_{n>0} a_n e^{-nt} e^{inx} \right|^2 \mathrm{d}t\,\mathrm{d}x$$

Proof.

Test observability inequality against $g(t,x) = \sum_{n>0} a_n e^{-nt} e^{inx}$:

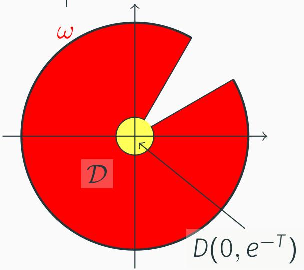
$$\sum_{n>0} |a_n|^2 e^{-2nT} \leq C \int_{[0,T]\times\omega} \left| \sum_{n>0} a_n e^{-nt} e^{inx} \right|^2 \mathrm{d}t\,\mathrm{d}x$$

- Chg of variables: $z = e^{-t+ix}$

$$|g|^2_{L^2([0,T]\times\omega)} = \int_{\mathcal{D}} \left| \sum_{n>0} a_n z^{n-1} \right|^2 \mathrm{d}\lambda(z)$$

- Polar coordinates:

$$|g(T,\cdot)|^2_{L^2(\mathbb{T})} \geq \pi^{-1} \int_{D(0,e^{-T})} \left| \sum_{n>0} a_n z^{n-1} \right|^2 \mathrm{d}\lambda(z)$$

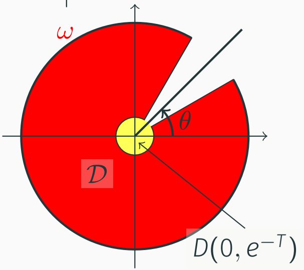


Proof.

Test observability inequality against $g(t,x) = \sum_{n>0} a_n e^{-nt} e^{inx}$:

$$\sum_{n>0} |a_n|^2 e^{-2nT} \leq C \int_{[0,T]\times\omega} \left| \sum_{n>0} a_n e^{-nt} e^{inx} \right|^2 \mathrm{d}t\,\mathrm{d}x$$

- Chg of variables: $z = e^{-t+ix}$
$$|g|^2_{L^2([0,T]\times\omega)} = \int_{\mathcal{D}} \left| \sum_{n>0} a_n z^{n-1} \right|^2 \mathrm{d}\lambda(z)$$
- Polar coordinates:
$$|g(T,\cdot)|^2_{L^2(\mathbb{T})} \geq \pi^{-1} \int_{D(0,e^{-T})} \left| \sum_{n>0} a_n z^{n-1} \right|^2 \mathrm{d}\lambda(z)$$



- Observability $\Rightarrow$ for every $p \in \mathbb{C}[X]$, $|p|_{L^2(D(0,e^{-T}))} \leq C|p|_{L^2(\mathcal{D})}$
- Untrue thanks to Runge's theorem (chose $p_k(z) \longrightarrow 1/z$ away from $\mathbb{C} \setminus e^{i\theta}\mathbb{R}_+$) $\square$

Fractional heat equation

- Fractional Laplace operator: $(-\Delta)^{\alpha} f = \mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}f(\xi))$
- Control system: $(\partial_t + (-\Delta)^{\alpha}) f(t,x) = 1_{\omega} u, \quad x \in \mathbb{R}$

Fractional heat equation

- Fractional Laplace operator: $(-\Delta)^{\alpha} f = \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F} f(\xi))$
- Control system: $(\partial_t + (-\Delta)^{\alpha}) f(t,x) = 1_{\omega} u, \quad x \in \mathbb{R}$

Theorem (Lack of null-controllability of the fractional heat equation)

Let $\alpha < 1/2$, $T > 0$, and ω a strict open subset of $\mathbb{R}$. The fractional heat equation

$$(\partial_t + (-\Delta)^{\alpha}) f = 1_{\omega} u$$

is not null-controllable on ω in time T.

Fractional heat: lack of null-controllability

$\Omega = \mathbb{R}$, $\omega = \{|x| > \epsilon\}$.

Proof.

- Controllability $\Leftrightarrow$ observability:
  $(\partial_t + (-\Delta)^{\alpha})g = 0 \implies |g(T,\cdot)|_{L^2(\Omega)} \leq |g|_{L^2([0,T]\times\omega)}$

Fractional heat: lack of null-controllability

$\Omega = \mathbb{R}$, $\omega = \{|x| > \epsilon\}$.

Proof.

- Controllability $\Leftrightarrow$ observability:
  $(\partial_t + (-\Delta)^{\alpha})g = 0 \implies |g(T,\cdot)|_{L^2(\Omega)} \leq |g|_{L^2([0,T]\times\omega)}$
- g_0 that is concentrated at 0: $g_0(x) = \qquad e^{-x^2/2h}$

Fractional heat: lack of null-controllability

$\Omega = \mathbb{R}$, $\omega = \{|x| > \epsilon\}$.

Proof.

- Controllability $\Leftrightarrow$ observability:
  $(\partial_t + (-\Delta)^{\alpha})g = 0 \implies |g(T,\cdot)|_{L^2(\Omega)} \leq |g|_{L^2([0,T]\times\omega)}$
- g_0 that is concentrated at 0: $g_0(x) =$ $e^{-x^2/2h+ix\xi_0/h}$

Fractional heat: lack of null-controllability

$\Omega = \mathbb{R}$, $\omega = \{|x| > \epsilon\}$.

Proof.

- Controllability $\Leftrightarrow$ observability:
  $(\partial_t + (-\Delta)^{\alpha})g = 0 \implies |g(T,\cdot)|_{L^2(\Omega)} \leq |g|_{L^2([0,T]\times\omega)}$
- g_0 that is concentrated at 0: $g_0(x) = \chi(hD_x - \xi_0)e^{-x^2/2h + ix\xi_0/h}$

$\Omega = \mathbb{R}$, $\omega = \{|x| > \epsilon\}$.

Proof.

- Controllability $\Leftrightarrow$ observability:
  $(\partial_t + (-\Delta)^{\alpha})g = 0 \implies |g(T,\cdot)|_{L^2(\Omega)} \leq |g|_{L^2([0,T]\times\omega)}$
- g_0 that is concentrated at 0: $g_0(x) = \chi(hD_x - \xi_0)e^{-x^2/2h + ix\xi_0/h}$

$$g(t,x) = c_h e^{ix\xi_0/h - x^2/2h} \int_{\mathbb{R}} \chi(\xi) e^{-(\xi - ix)^2/2h - t|\xi+\xi_0|^{2\alpha}/h^{2\alpha}} \mathrm{d}\xi$$

- Saddle point method:

$$g(t,x) = \mathcal{O}\left(\frac{1}{|x|^\infty} e^{-ct/h}\right) \qquad |x| > \epsilon$$

$$g(t,x) = e^{ix\xi_0/h - x^2/2h - \mathcal{O}(h^{-2\alpha})} \qquad |x| < \frac{\xi_0}{4}$$

□

Results on the Grushin equation

Grushin equation

$$(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 1_{\omega}u(t,x,y),\ x \in \mathbb{R}, y \in \mathbb{T}$$

Grushin equation

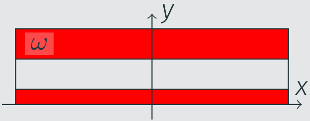
$$(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = 1_\omega u(t,x,y),\ x \in \mathbb{R}, y \in \mathbb{T}$$

«Embedding» of the half-heat in the Grushin equation

- For $n \in \mathbb{N}$, $e^{-nx^2/2+iny}$ eigenfunction, with eigenvalue n
- Particular solutions: $g(t,x,y) = \sum_{n>0} a_n e^{-nt-nx^2/2+iny}$
- In y-variable: similar to solutions of the half-heat

Control of the Grushin equation

Theorem (Grushin equation on horizontal band)

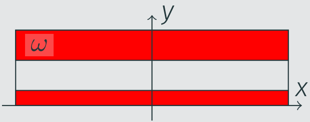


$\omega = \mathbb{R} \times \omega_y$

Never null-controllable

Control of the Grushin equation

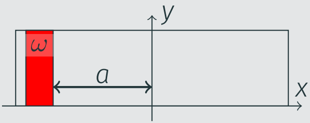
Theorem (Grushin equation on horizontal band)



$\omega = \mathbb{R} \times \omega_y$

Never null-controllable

Theorem (Beauchard-Dardé-Ervedoza 2018)

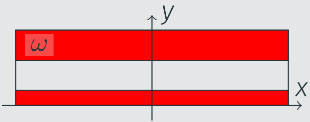


$\omega = (a, b) \times \mathbb{T}$

Null-controllable on ω *iff* $T > a^2/2$.

Control of the Grushin equation

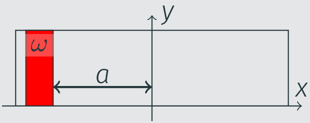
Theorem (Grushin equation on horizontal band)



$\omega = \mathbb{R} \times \omega_y$

Never null-controllable

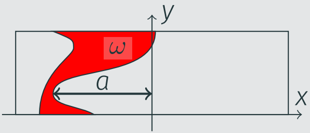
Theorem (Beauchard-Dardé-Ervedoza 2018)



$\omega = (a, b) \times \mathbb{T}$

Null-controllable on ω *iff* $T > a^2/2$.

Theorem (Duprez-K 2018)



$\omega = \{\gamma_1(y) < x < \gamma_2(y)\}$, $a = \max(\sup(\gamma_2^-), \sup(\gamma_1^+))$

Null-controllable on ω *if* $T > a^2/2$

Not null-controllable on ω *if* $T < a^2/2$.

Conclusion

What have we learned ?

- Heat equation: always null-controllable

What have we learned ?

- Heat equation: always null-controllable
- Situation much more complicated for degenerate parabolic equations than for heat equation
- Special cases only/ad-hoc methods
- Mystery: minimal time see everything between the degeneracy and the control region

What have we learned ?

- Heat equation: always null-controllable
- Situation much more complicated for degenerate parabolic equations than for heat equation
- Special cases only/ad-hoc methods
- Mystery: minimal time see everything between the degeneracy and the control region

That's all folks!