Null-controllability of parabolic-transport systems

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Control in Time of Crisis

Introduction

Ω domain of $\mathbb{R}^n$, ω an open subset of Ω and $T > 0$.

Definition (Null-controllability of the heat equation on ω in time T)

For every initial condition $f_0 \in L^2(\Omega)$, there exists a control $u \in L^2([0, T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = 1_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

satisfies $f(T, \cdot) = 0$ on Ω.

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For every initial condition $f_0 \in L^2(\Omega)$, there exists a control $u \in L^2([0,T] \times \omega)$ such that the solution f of:

$$\partial_t f - \Delta f = \mathbb{1}_\omega u, \quad f_{|\partial\Omega} = 0, \quad f(0) = f_0$$

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Theorem (Null-controllability of the heat equation (Lebeau & Robbiano 1995, Fursikov & Imanuvilov 1996))

Ω a C^2 connected bounded open subset of $\mathbb{R}^n$, ω a non-empty open subset of Ω, and $T > 0$. The heat equation is null-controllable on ω in time T.

The equation:

$$\partial_t f(t,x) + A\partial_x f(t,x) - B\partial_x^2 f(t,x) + Kf(t,x) = 1_\omega u(t,x), \quad (t,x) \in [0,+\infty[\times \mathbb{T}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix},\ D + D^* \text{ positive-definite ;} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\ A_{11} = A_{11}^*.$$

Coupling between parabolic and transport equations

$$f = \begin{pmatrix} f_h \\ f_p \end{pmatrix},\ \begin{cases} (\partial_t + A_{11}\partial_x + K_{11})f_h(t,x) + (A_{12}\partial_x + K_{12})f_p(t,x) = 1_\omega u_h(t,x) \\ (\partial_t - D\partial_x^2 + A_{22}\partial_x + K_{22})f_p(t,x) + (A_{21}\partial_x + K_{21})f_h(t,x) = 1_\omega u_p(t,x) \end{cases}$$

Question
For every, $f_0 \in L^2(\mathbb{T}, \mathbb{C}^d)$ does there exist $u \in L^2([0,T] \times \omega, \mathbb{C}^d)$ such that $f(T,\cdot) = 0$? What if we ask for $u_h = 0$ (or $u_p = 0$) ?

The results

Theorem (Beauchard-K-Le Balc'h 2019)

ω an open interval of $\mathbb{T}$.

$$T^* = \frac{2\pi - \operatorname{length}(\omega)}{\min_{\mu \in \operatorname{Sp}(A_{11})} |\mu|}$$

Then

1. *the system is not null-controllable on ω in time $T < T^*$,*
2. *the system is null-controllable on ω in time $T > T^*$.*

Minimal time = minimal time for the transport equation
In the case

$$\partial_t f_h + A_{11} \partial_x f_h = u_h \mathbf{1}_\omega$$

Free solutions = sums of waves travelling at speed $\mu_k \in \operatorname{Sp}(A_{11})$.

Theorem (Hyperbolic control, $D = I$ and $K = 0$, Beauchard-K-Le Balc'h 2020)

$$f = \begin{pmatrix} f_h \\ f_p \end{pmatrix}, \quad \begin{cases} (\partial_t + A_{11}\partial_x) f_h(t,x) + A_{12}\partial_x f_p(t,x) = \mathbf{1}_\omega u_h(t,x) \\ (\partial_t - \partial_x^2 + A_{22}\partial_x) f_p(t,x) + A_{21}\partial_x f_h(t,x) = 0 \end{cases}$$

Controllability in time $T > T^*$ for initial conditions with zero average iff $\mathrm{Vect}\{A_{22}^i A_{21} v, i \in \mathbb{N}, v \in \mathbb{C}^{d_h}\} = \mathbb{C}^{d_p}$

Theorem (Parabolic control and $K = 0$, Beauchard-K-Le Balc'h 2020)

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Controllability in time $T > T^*$ for initial conditions in H^{d_1+1} with zero average if $\mathrm{Vect}\{A_{11}^i A_{12} v, i \in \mathbb{N}, v \in \mathbb{C}^{d_p}\} = \mathbb{C}^{d_h}$.

Navier-Stokes
ρ: fluid density. v: fluid velocity. $a, \gamma, \mu > 0$.

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 1_\omega u_1(t,x) \text{ on } [0,T]\times\mathbb{T} \\ \rho(\partial_t v + v\partial_x v) + \partial_x(a\rho^\gamma) - \mu\partial_x^2 v = 1_\omega u_2(t,x) \text{ on } [0,T]\times\mathbb{T} \end{cases}$$

Linearization around a stationnary state $(\bar\rho, \bar v) \in \mathbb{R}_+^* \times \mathbb{R}^*$:

$$\begin{cases} \partial_t \rho + \bar v\partial_x\rho + \bar\rho\partial_x v = 1_\omega u_1(t,x) \text{ sur } [0,T]\times\mathbb{T} \\ \partial_t v + \bar v\partial_x v + a\bar\rho^{\gamma-2}\partial_x\rho - \frac{\mu}{\rho}\partial_x^2 v = 1_\omega u_2(t,x) \text{ on } [0,T]\times\mathbb{T} \end{cases}$$

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- [Ervedoza-Guerrero-Glass-Puel 2012]: equation posed on $(0,L)$, boundary control acting on (ρ, v) in time $T > L/|\bar v|$
- [Chowdhury-Mitra-Ramaswamy-Renardy 2014]: velocity control in time $T > 2\pi/|\bar v|$ for the initial conditions $(\rho_0, v_0) \in H^1 \times L^2$.
- [Beauchard-K-Le Balc'h 2020] with $A = \begin{pmatrix} \bar v & \bar\rho \\ a\bar\rho^{\gamma-2} & \bar v \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & \mu/\rho \end{pmatrix}$: velocity control, in time $T > (2\pi - \text{length}(\omega))/|\bar v|$ for initial conditions in $H^2 \times H^2$.

(Idea of the) proof

Fourier components

$$(-B\partial_x^2 + A\partial_x)Xe^{inx} = n^2\left(B + \frac{i}{n}A\right)Xe^{inx}$$

Spectrum of $-B\partial_x^2 + A\partial_x$

$$\mathrm{Sp}(-B\partial_x^2 + A\partial_x) = \left\{n^2\,\mathrm{Sp}\left(B + \frac{i}{n}A\right)\right\}$$

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Perturbation theory

λ_{nk} eigenvalue of $B + \frac{i}{n}A$. λ_k eigenvalue of B: $\lambda_{nk} \to \lambda_k \in \mathrm{Sp}(B)$

- If $\lambda_k \neq 0$, $n^2\lambda_{nk} \underset{n\to+\infty}{\sim} n^2\lambda_k$: parabolic frequencies
- If $\lambda_k = 0$, $n^2\lambda_{nk} \underset{n\to+\infty}{\sim} in\mu_k$: hyperbolic frequencies

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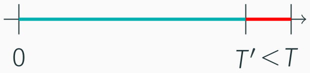
- If $\lambda_k = 0$, $n^2\lambda_{nk} \underset{n\to+\infty}{\sim} in\mu_k$: hyperbolic frequencies
- Free solutions: $= \sum X_{nk}e^{inx-n^2\lambda_{nk}t} \approx \sum_{\text{parabolic}} X_{nk}e^{inx-n^2\lambda_k t} + \sum_{\text{hyperbolic}} X_{nk}e^{inx-in\mu_k t}$
- Well-posed if $\Re(\lambda_k) > 0$ and $\mu_k \in \mathbb{R}$
- Not null-controllable in small time

Decouple and control

- 

Decouple and control

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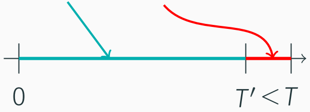
- 

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- Step 1: null-controllability of a parabolic equation in time $T - T' > 0$
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- Step 1: null-controllability of a parabolic equation in time $T - T' > 0$
- Step 2: exact controllability of a perturbed transport equation in time T'. Ok if $T' > T^*$.
- Deal the finite dimensional subspaces that are left: compactness-uniqueness

Novelty with respect to Lebeau-Zuazua

Systems of arbitrary size

- Strategy as described until now: Lebeau-Zuazua (1998) for linear systems of thermoelasticity (coupled heat-wave)
- Our work: generalize for systems of arbitrary size

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- Strategy as described until now: Lebeau-Zuazua (1998) for linear systems of thermoelasticity (coupled heat-wave)
- Our work: generalize for systems of arbitrary size
- Difficulty: eigenvalues and eigenvectors $B + \frac{i}{n}A$ can behave badly as $n \to +\infty$
- Solution: don't use eigenvectors nor eigenvalues
- We use *total eigenprojections*: sum of eigenprojections associated to eigenvalues that are close to each other (Kato's perturbation theory...)

$$-\frac{1}{2i\pi}\oint_\Gamma (M - z)^{-1}\,\mathrm{d}z = \text{Eigenprojection on eigenspaces associated to eigenvalues of } M \text{ lying inside } \Gamma$$

- Kato's *reduction process*

Conclusion

Parabolic-transport $\simeq$ transport

- null-controllable iff transport is controllable

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Open problems

- domain other that $\mathbb{T}$?
- less controls than equations?
- non-constant coefficient?
- unique continuation?
- ...

That’s all folks!