An inequality on operators on polynomials

Armand Koenig

17 October 2022

Workshop on Control Problems 2022

Introduction

The problem
Estimating

$$\Big\| \underbrace{\sum_n \gamma(n) a_n z^n}_{\gamma(z\partial_z)(\sum_n a_n z^n)} \Big\| \leq C \Big\| \sum_n a_n z^n \Big\|$$

if $\gamma(n) = n$ $\quad \gamma(z\partial_z) P = (z\partial_z) P$

for useful norms.

The problem
Estimating

$$\|\underbrace{\sum_n \gamma(n) a_n z^n}_{\gamma(z\partial_z)(\sum_n a_n z^n)}\| \leq C \|\sum_n a_n z^n\|$$

for useful norms.

Simple hilbertian idea
Orthognality of $z \mapsto z^n$ on $D(0, R)$:

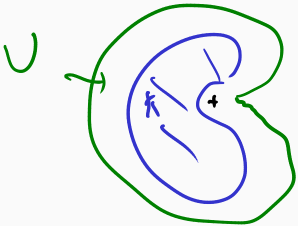
$$\|\gamma(z\partial_z)P\|_{L^2(D(0,R))} \leq \sup_n |\gamma_n| \|P\|_{L^2(D(0,R))}.$$

What about L^p-norm on domain which are not a disk?

Theorem

Let $\gamma\colon \{z \in \mathbb{C},\ \Re(z) > 0\} \to \mathbb{C}$ holomorphic and bounded. Let $K \subset \mathbb{C}$ compact. Let U be an open neighbourhood of K that is star-shaped with respect to 0. There exists $C > 0$ such that for every polynomial P,

$$\|\gamma(z\partial_z)P\|_{L^\infty(K)} \leq C\|P\|_{L^\infty(U)}.$$



Idea of the proof

$\gamma(z\partial_z)$ is a convolution operator

Theorem

Assume $\sup_n |\gamma(n)| < +\infty$. *Let*

$$K_\gamma(\zeta) = \sum_n \gamma(n)\zeta^n.$$

Then, for $|z| < R$

$$\gamma(z\partial_z)P(z) = \frac{1}{2i\pi}\oint_{\partial D(0,R)} P(\zeta)K_\gamma\left(\frac{z}{\zeta}\right)\frac{\mathrm{d}\zeta}{\zeta}.$$

Proof. $P(z) = \sum a_n z^n$ Cauchy: $a_n = \frac{1}{2i\pi}\oint_{\partial D(0,R)} \frac{P(\zeta)}{\zeta^{n+1}} d\zeta$

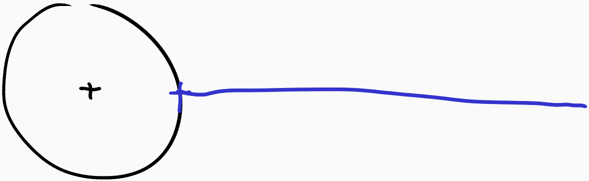
$$\gamma(z\partial_z)P(z) = \sum \gamma(n) a_n z^n = \frac{1}{2i\pi}\oint_{\partial D(0,R)} \sum \gamma(n)\frac{P(\zeta)}{\zeta^{n+1}} z^n \, d\zeta$$

$K_\gamma\left(\frac{z}{\zeta}\right)$

□

Theorem (Lindelöf (1905))

Assume $\gamma\colon \{\Re(z) > 0\} \to \mathbb{C}$ *is holomorphic and bounded. Then* $K_\gamma(\zeta) = \sum_n \gamma(n)\zeta^n$ *extends as a holomorphic function on* $\mathbb{C} \setminus [1, +\infty)$.

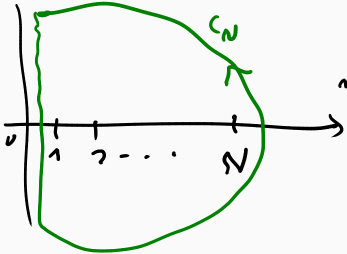


Theorem (Lindelöf (1905))

Assume $\gamma\colon \{\Re(z) > 0\} \to \mathbb{C}$ is holomorphic and bounded. Then $K_\gamma(\zeta) = \sum_n \gamma(n)\zeta^n$ extends as a holomorphic function on $\mathbb{C} \setminus [1, +\infty)$.

Idea of the proof.

$$K_\gamma(\zeta) = \int_{\frac{1}{2}+i\infty}^{\frac{1}{2}-i\infty} \frac{\gamma(z)\zeta^z}{e^{2i\pi z} - 1}\,\mathrm{d}z$$



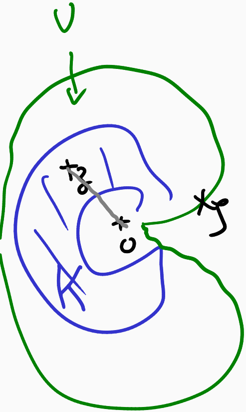
$$\sum_{n=1}^{N} \gamma(n)\zeta^n = \int_{C_N} \frac{\gamma(z)\zeta^z}{e^{2i\pi z} - 1}\,dz$$

take $N \to +\infty$

Proof of the estimate.

$$\gamma(z\partial_z)P(z) = \frac{1}{2i\pi}\oint_{\partial D(0,R)\ \to\ \partial U} P(\zeta)K_\gamma\left(\frac{z}{\zeta}\right)\frac{\mathrm{d}\zeta}{\zeta}.$$

K_γ holomorphic on $\mathbb{C}\setminus[1,+\infty)$.



U star-shaped with respect to 0

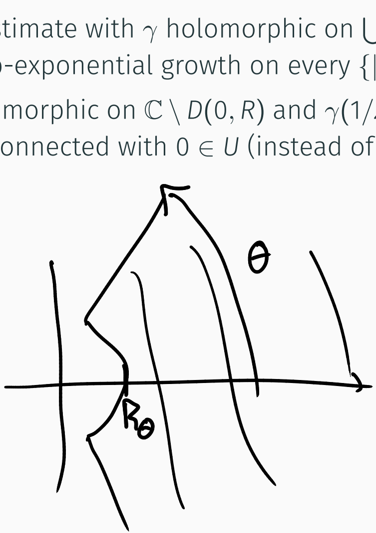
$\Rightarrow \frac{z}{\zeta} \notin [1,+\infty)$

$$|\gamma(z\partial_z)P(z)| \leq \frac{1}{2\pi}\,\mathrm{length}(\partial U)\,\|P\|_{L^\infty(\partial U)} \times \|K_\gamma\|_{L^\infty(\text{some compact})}\frac{1}{d(0,\partial U)} \quad \square$$

Tricks

Variants

- Same estimate with γ holomorphic on $\bigcup_{\theta\in(0,\pi/2)}\{|\arg(z)|<\theta\}\setminus D(0,R_\theta)$, with sub-exponential growth on every $\{|\arg(z)|<\theta\}$
- If γ holomorphic on $\mathbb{C}\setminus D(0,R)$ and $\gamma(1/z)$ holomorphic at 0, version with U simply connected with $0\in U$ (instead of U star-shaped with respect to 0).



Variants

- Same estimate with γ holomorphic on $\bigcup_{\theta\in(0,\pi/2)}\{|\arg(z)|<\theta\}\setminus D(0,R_\theta)$, with sub-exponential growth on every $\{|\arg(z)|<\theta\}$
- If γ holomorphic on $\mathbb{C}\setminus D(0,R)$ and $\gamma(1/z)$ holomorphic at 0, version with U simply connected with $0\in U$ (instead of U star-shaped with respect to 0).

Following the constants

- Everything is continuous with respect to their natural topologies
- $\begin{array}{ccc} L^\infty(\mathbb{C}_+)\cap\mathcal{O}(\mathbb{C}_+) & \to & \mathcal{L}(L^\infty(U)\cap\mathcal{O}(U),\, L^\infty(K)\cap\mathcal{O}(K)) \\ \gamma & \longmapsto & \gamma(z\partial_z) \end{array}$ is continuous
- $\|\gamma(z,z\partial_z)P\|_{L^\infty(K)}\leq\sup\limits_{\zeta\in K}\|\gamma(\zeta,z\partial_z)P\|_{L^\infty(K)}\leq C\|P\|_{L^\infty(U)}$ (assuming uniform bounds on $\gamma(\zeta,z)$)

Applications

Control of the Baouendi-Grushin equation
See Jérémi Dardé's talk

Eigenfunction of $-z^2\partial_x^2 + x^2$ on $(-1,1)$ (with application to the control of a Kolmogorov-type equation)

- $\theta_0 \in (0, \pi/2)$, limit $|z| \to 0$, $|\arg(z)| < \theta_0$
- First eigenvalue: $\lambda_z = z + 4\sqrt{\frac{z}{\pi}} e^{-1/z}(1 + O(z)) = z(1 + 2\rho_z)$
- Asymptotics for the first eigenfunction $g_z(x)$?

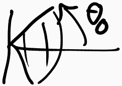
$$(\partial_t - \partial_v^2 + v^2 \partial_x) g(t, v, x) = 0$$

$$\Big\{ \mathcal{F}_x$$

$$\xi^{-2}$$

$$\partial_t - \partial_v^2 - i\xi v^2$$

Eigenfunction of $-z^2\partial_x^2 + x^2$ on $(-1,1)$ (with application to the control of a Kolmogorov-type equation)

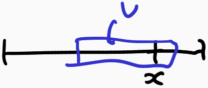


- $\theta_0 \in (0, \pi/2)$, limit $|z| \to 0$, $|\arg(z)| < \theta_0$
- First eigenvalue: $\lambda_z = z + 4\sqrt{\frac{z}{\pi}}e^{-1/z}(1 + O(z)) = z(1 + 2\rho_z)$
- Asymptotics for the first eigenfunction $g_z(x)$?
- Write equation for $w_z(x) = e^{x^2/2z}g_z(x)$, solve it as a power series:

$$w_z(x) = 1 + \rho_z \sum_{n>1} \underbrace{\left(-\frac{1}{2n}\frac{4^n(n!)^2}{(2n)!}\prod_{k=1}^{n-1}\left(1 - \frac{\rho_z}{2k}\right)\right)}_{\gamma_z(n)} \frac{1}{n!}\left(\frac{x^2}{z}\right)^n.$$

$$= 1 + \rho_z \gamma_z(x\partial_x)(e^{x^2/z} - 1)$$

$$= 1 + O(e^{-1/z + x^2/z + \delta/z})$$



$$g_z(x) = e^{-x^2/2z}(1 + O(e^{-c/z}))$$

That's all folks!