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Karine Beauchard, Armand Koenig, & Kévin Le Balc'h

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NULL-CONTROLLABILITY OF LINEAR PARABOLIC-TRANSPORT SYSTEMS

by Karine Beauchard, Armand Koenig & Kévin Le Balc'h

Abstract. — Over the past two decades, the controllability of several examples of parabolic-hyperbolic systems has been investigated. The present article is the beginning of an attempt to find a unified framework that encompasses and generalizes the previous results. We consider constant coefficients parabolic-transport systems with coupling of order zero and one, with a locally distributed control in the source term, posed on the one-dimensional torus. We prove the null-controllability, in optimal time (the one expected because of the transport component) when there is as many controls as equations. When the control acts only on the transport (resp. parabolic) component, we prove an algebraic necessary and sufficient condition, on the coupling term, for the null-controllability. The whole study relies on a careful spectral analysis, based on perturbation theory. For high frequencies, the spectrum splits into a parabolic part and a hyperbolic part. The negative controllability result in small time is proved on solutions localized on high hyperbolic frequencies, that solve a pure transport equation up to a compact term. The positive controllability result in large time is proved by projecting the dynamics onto three eigenspaces associated to hyperbolic, parabolic and low frequencies, that defines three weakly coupled systems.

Résumé (Contrôlabilité à zéro des systèmes paraboliques-transport linéaires couplés)

Depuis une vingtaine d'années, la contrôlabilité de plusieurs exemples de systèmes paraboliques-hyperboliques couplés a été étudiée. Nous initions dans cet article une recherche d'un cadre qui contient et généralise les résultats déjà existants. Nous considérons des systèmes paraboliques-transport, à coefficients constants, couplés par des termes d'ordre 0 et 1, posés sur le tore de dimension 1, et avec contrôle interne localisé sur un ouvert du tore. Nous démontrons la contrôlabilité à zéro de ces systèmes en temps optimal (celui attendu en raison des composantes de transport) lorsqu'on contrôle toutes les équations. Lorsque le contrôle agit uniquement sur les composantes hyperboliques (resp. paraboliques), nous démontrons une condition nécessaire et suffisante pour la contrôlabilité à zéro, cette condition étant de type Kalman et portant sur le terme de couplage. Cette étude repose sur une analyse spectrale, elle-même basée sur la théorie perturbative : en hautes fréquences, le spectre se sépare en une branche hyperbolique et une branche parabolique. Le résultat de non-contrôlabilité en temps petit est démontré en construisant des solutions de transport approchées, localisées en hautes fréquences. Le résultat de contrôle en temps grand est démontré en projetant la dynamique sur trois espaces stables, associés respectivement aux hautes fréquences hyperboliques, hautes fréquences paraboliques et basses fréquences, ce qui définit trois systèmes faiblement couplés.

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Contents

1. Introduction 744
2. Preliminary results 753
3. Obstruction to the null-controllability in small time 760
4. Large time null-controllability 762
5. Hyperbolic control: coupling of order zero 782
6. Hyperbolic control: coupling of order one 787
7. Parabolic control 791
Appendix. Pure transport solutions are not enough to disprove the observability inequality 800
References 801

1. Introduction

1.1. Parabolic-transport systems. — We consider the linear control system

$$\text{(Sys)}\qquad \begin{cases} \partial_t f - B\partial_x^2 f + A\partial_x f + Kf = Mu1_\omega & \text{in } (0,T)\times\mathbb{T},\\ f(0,\cdot) = f_0 & \text{in } \mathbb{T},\end{cases}$$

where

– $T>0$, $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$, ω is a nonempty open subset of $\mathbb{T}$, $d\in\mathbb{N}^*$, $m\in\{1,\dots,d\}$, $A,B,K\in\mathbb{R}^{d\times d}$, $M\in\mathbb{R}^{d\times m}$,
– the state is $f\colon [0,T]\times\mathbb{T}\to\mathbb{R}^d$,
– the control is $u\colon [0,T]\times\mathbb{T}\to\mathbb{R}^m$.

We assume

$$\text{(H.1)}\qquad d = d_1 + d_2 \text{ with } 1\leqslant d_1 < d,\ 1\leqslant d_2 < d,$$

$$\text{(H.2)}\qquad B = \begin{pmatrix} 0 & 0\\ 0 & D\end{pmatrix},\ \text{with } D\in\mathbb{R}^{d_2\times d_2},$$

$$\text{(H.3)}\qquad \Re(\mathrm{Sp}(D))\subset(0,\infty).$$

Introducing the analogue block decomposition for the $d\times d$ matrices A and K, the $d\times m$ matrix M and the function f,

$$A = \begin{pmatrix} A' & A_{12}\\ A_{21} & A_{22}\end{pmatrix},\quad K = \begin{pmatrix} K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix},\quad M = \begin{pmatrix} M_1\\ M_2\end{pmatrix},\quad f(t,x) = \begin{pmatrix} f_1(t,x)\\ f_2(t,x)\end{pmatrix},$$

we see that the system (Sys) couples a transport equation on f_1 with a parabolic equation on f_2:[(1)]

$$(1)\qquad \begin{cases} (\partial_t + A'\partial_x + K_{11})f_1 + (A_{12}\partial_x + K_{12})f_2 = M_1 u1_\omega & \text{in } (0,T)\times\mathbb{T},\\ \big(\partial_t - D\partial_x^2 + A_{22}\partial_x + K_{22}\big)f_2 + (A_{21}\partial_x + K_{21})f_1 = M_2 u1_\omega & \text{in } (0,T)\times\mathbb{T},\\ (f_1,f_2)(0,\cdot) = (f_{01},f_{02}) & \text{in } \mathbb{T}.\end{cases}$$

(1) As we will see, the submatrix A' of A is of considerable importance. We denote it by A' (instead of A_{11}) to underline the special role it plays.

We make the following hypothesis on the matrix A'

$$A' \text{ is diagonalizable with } \operatorname{Sp}(A') \subset \mathbb{R}. \tag{H.4}$$

We will prove later, with vector valued Fourier series and a careful spectral analysis, that for every $f_0 \in L^2(\mathbb{T}, \mathbb{C}^d)$ and $u \in L^2((0,T) \times \mathbb{T}, \mathbb{C}^m)$, there exists a unique solution $f \in C^0([0,T], L^2(\mathbb{T})^d)$ of (Sys) (see Section 2.2.3). In this article, we are interested in the null-controllability of (Sys).

Definition 1. — The system (Sys) is null-controllable on ω in time T if for every $f_0 \in L^2(\mathbb{T}; \mathbb{C}^d)$, there exists a control $u \in L^2((0,T) \times \mathbb{T}, \mathbb{C}^m)$ supported on $[0,T] \times \omega$ such that the solution f of (Sys) satisfies $f(T, \cdot) = 0$.

For a given ω, it might happen that the system (Sys) is null-controllable for some $T_1 > 0$ but not for some other $T_2 > 0$ (and we will see that it usually happens). We call the *minimal time of null-controllability* (on ω) the infimum of the $T > 0$ such that the system (Sys) is null-controllable on ω in time T. If we note $T_{\min}$ this minimal time of null-controllability, that may *a priori* take any value in $[0, +\infty]$, the system (Sys) is null-controllable on ω if $T > T_{\min}$ and it is not null-controllable on ω if $T < T_{\min}$. We aim at

– identifying the minimal time for null-controllability, depending on the matrices A, B, K and the domain ω,

– controlling the system with a small number of controls $m < d$,

– understanding the influence of the algebraic structure (A, B, K, M) on the above properties.

1.2. Statement of the results

1.2.1. *Control on any component, minimal time.* — Our first result identifies the minimal time, when the control acts on each of the d equations.

Theorem 2. — *We assume that ω is a strict open subset of $\mathbb{T}$. We also assume* (H.1)–(H.4) *and that the control matrix is $M = I_d$ (and so $m = d$). We define*[2]

$$\ell(\omega) := \sup\{|I|;\ I \text{ connected component of } \mathbb{T} \setminus \omega\}, \tag{2}$$

$$\mu_* = \min\{|\mu|;\ \mu \in \operatorname{Sp}(A')\},$$

and

$$T^* = \begin{cases} \ell(\omega)/\mu_* & \text{if } \mu_* > 0, \\ +\infty & \text{if } \mu_* = 0. \end{cases} \tag{3}$$

Then

(i) *the system* (Sys) *is not null-controllable on ω in time $T < T^*$,*

(ii) *the system* (Sys) *is null-controllable on ω in any time $T > T^*$.*

[2] If $I \subset \mathbb{R}$ is measurable, we note $|I|$ its Lebesgue measure.

In particular, when ω is an interval of $\mathbb{T}$ and $\mu_* > 0$, then the minimal time for null-controllability is $T^* = (2\pi - |\omega|)/\mu_*$.

Actually, the controls may be more regular than in Definition 1: we construct controls of the form $u=(u_1,u_2)$ where $u_1\in L^2((0,T)\times\omega)^{d_1}$ and $u_2 \in C_c^\infty((0,T)\times\omega)^{d_2}$.

The proof of Theorem 2 relies on a spectral decomposition: for high frequencies, the spectrum splits into a parabolic part and a hyperbolic part.

The negative result in time $T < T^*$ is expected, because of the transport component of the system, but its proof does require some care. Indeed, because of the coupling with a parabolic component, in general, there does not exist pure transport solutions to the system (Sys) that are concentrated outside $(0,T) \times \omega$ (see the appendix for more precision).

The proof of the positive result, in time $T > T^*$ relies on an adaptation, to systems with arbitrary size, of the strategy introduced by Lebeau and Zuazua [22] to control the system of linear thermoelasticity, that couples a scalar heat equation and a scalar wave equation. By projecting the dynamics onto appropriate eigenspaces, the system is decomposed into three weakly coupled systems. The first one behaves like a transport system, its controllability is handled by hyperbolic methods from [1]. The second one behaves like a parabolic system, its controllability is handled by the Lebeau-Robbiano method. The third one, associated to low frequencies, has finite dimension; its controllability is handled by a compactness/uniqueness argument.

The null-controllability of the system (Sys) in time $T = T^*$ is an open problem.

1.2.2. *Control on the hyperbolic component.* — Our second result concerns controls acting on the whole transport component, $M_1 = I_{d_1}$, but not on the parabolic component, $M_2 = 0$. To get an aesthetic necessary and sufficient algebraic condition for null-controllability, we also assume that the diffusion is given by $D = I_{d_2}$, the coupling is realized exclusively by the transport term $A_{21}\partial_x f_1$, i.e., $K_{21} = 0$ and there is no zero order term in the parabolic dynamics, i.e., $K_{22} = 0$, which corresponds to the system

$$\begin{cases} (\partial_t + A'\partial_x + K_{11})f_1 + (A_{12}\partial_x + K_{12})f_2 = u_1 1_\omega & \text{in } (0,T)\times\mathbb{T}, \\ \big(\partial_t - \partial_x^2 + A_{22}\partial_x\big)f_2 + A_{21}\partial_x f_1 = 0 & \text{in } (0,T)\times\mathbb{T}. \end{cases} \tag{4}$$

By integrating with respect to the space variable the second equation of (4), we see that, for being steered to zero, an initial condition $f_0 = (f_{01}, f_{02}) \in L^2(\mathbb{T})^{d_1} \times L^2(\mathbb{T})^{d_2}$ has to satisfy

$$\int_{\mathbb{T}} f_{02}(x)\,\mathrm{d}x = 0. \tag{5}$$

For any vector subspace E of $L^1(\mathbb{T})$ we denote by E_{m} the vector subspace made of functions $f \in E$ with zero mean value, i.e., $\int_{\mathbb{T}} f(x)dx = 0$.

Theorem 3. — *We assume* (H.1)–(H.4)*,* $D = I_{d_2}$ $m = d_1$*,* $M_1 = I_{d_1}$*,* $M_2 = 0$*,* $K_{21} = 0$ *and* $K_{22} = 0$*. Let* T^* *be defined by* (3)*. The following statements are equivalent:*

– *For every $T > T^*$ and $f_0 = (f_{01}, f_{02}) \in L^2(\mathbb{T})^{d_1} \times L^2_{\mathrm{m}}(\mathbb{T})^{d_2}$, there exists $u_1 \in L^2((0,T)\times\omega)^{d_1}$ such that the solution of (4) satisfies $f(T) = 0$.*

– *The pair of matrices (A_{22}, A_{21}) satisfies the Kalman rank condition:*

$$\mathrm{Span}\{A_{22}^j A_{21} X_1 ; X_1 \in \mathbb{C}^{d_1}, 0 \leqslant j \leqslant d_2 - 1\} = \mathbb{C}^{d_2}. \tag{6}$$

With the same proof, similar statements can be proved for the following systems:

$$\begin{cases} (\partial_t + A'\partial_x + K_{11}) f_1 + (A_{12}\partial_x + K_{12}) f_2 = u_1 1_\omega & \text{in } (0,T)\times\mathbb{T}, \\ \big(\partial_t - \partial_x^2 + K_{22}\big) f_2 + K_{21} f_1 = 0 & \text{in } (0,T)\times\mathbb{T}, \end{cases} \tag{7}$$

with arbitrary initial conditions $f_0 \in L^2(\mathbb{T})^d$ and Kalman rank condition on (K_{22}, K_{21}) (see Section 5),

$$\begin{cases} (\partial_t + A'\partial_x + K_{11}) f_1 + (A_{12}\partial_x + K_{12}) f_2 = u_1 1_\omega & \text{in } (0,T)\times\mathbb{T}, \\ \big(\partial_t - \partial_x^2 + A_{22}\partial_x\big) f_2 + K_{21} f_1 = 0 & \text{in } (0,T)\times\mathbb{T}, \end{cases} \tag{8}$$

with arbitrary initial conditions $f_0 \in L^2(\mathbb{T})^d$ and Kalman rank condition on (A_{22}, K_{21}),

$$\begin{cases} (\partial_t + A'\partial_x + K_{11}) f_1 + (A_{12}\partial_x + K_{12}) f_2 = u_1 1_\omega & \text{in } (0,T)\times\mathbb{T}, \\ \big(\partial_t - \partial_x^2 + K_{22}\big) f_2 + A_{21}\partial_x f_1 = 0 & \text{in } (0,T)\times\mathbb{T}, \end{cases} \tag{9}$$

with initial conditions $f_0 \in L^2(\mathbb{T})^d$ satisfying (5) and Kalman rank condition on (K_{22}, A_{21}).

If $T < T^*$, there exists $f_0 = (f_{01}, f_{02}) \in L^2(\mathbb{T})^{d_1} \times L^2_{\mathrm{m}}(\mathbb{T})^{d_2}$, such that, for any $u_1 \in L^2((0,T)\times\omega)^{d_1}$ the solution of (4) satisfies $f(T) \neq 0$. This result is not a consequence of Theorem 2 because this statement does not ensure that the counterexample f_0 has zero mean value on the second component. However, it can be proved by the same strategy as the negative result in time $T < T^*$ in Theorem 2, see Remark 17 for details.

The proof of the controllability of (4) uses two ingredients. The first ingredient is a strengthened version of Theorem 2 with smoother controls, more precisely, the associated observability inequality with observation of negative Sobolev norms of the parabolic component. The second ingredient is a cascade structure (or Brunovski form) of the system (4) ensured by the Kalman condition, to eliminate the observation of the parabolic component.

Proving an algebraic necessary and sufficient condition for null-controllability of (Sys), involving both matrices D, A and K is an open problem. In the context of parabolic systems, this difficulty already appeared, see [4] and [5].

1.2.3. *Control on the parabolic component*. — Our third result concerns controls acting on the whole parabolic component, $M_2 = I_{d_2}$, but not on the hyperbolic component of the system, $M_1 = 0$. To get an aesthetic necessary and sufficient condition for null-controllability, we also assume that the coupling is realized exclusively by the transport term $A_{12}\partial_x f_2$, i.e., $K_{12} = 0$, and there is no zero order term in the hyperbolic dynamics, i.e., $K_{11}=0$. This corresponds to the system

$$\begin{cases}(\partial_t + A'\partial_x)f_1 + A_{12}\partial_x f_2 = 0 & \text{in } (0,T)\times\mathbb{T},\\ \big(\partial_t - D\partial_x^2 + A_{22}\partial_x + K_{22}\big)f_2 + (A_{21}\partial_x + K_{21})f_1 = u_2 1_\omega & \text{in } (0,T)\times\mathbb{T},\\ (f_1,f_2)(0,\cdot) = (f_{01},f_{02}) & \text{in } \mathbb{T}.\end{cases} \tag{10}$$

By integrating with respect to the space variable the first equation of (10), we see that, for being steered to zero, an initial condition $f_0=(f_{01},f_{02})\in L^2(\mathbb{T})^{d_1}\times L^2(\mathbb{T})^{d_2}$ has to satisfy

$$\int_{\mathbb{T}} f_{01}(x)\,\mathrm{d}x = 0 \tag{11}$$

i.e., $f_0=(f_{01},f_{02})\in L^2_{\mathrm{m}}(\mathbb{T})^{d_1}\times L^2(\mathbb{T})^{d_2}$.

We need to adapt the notion of null-controllability, because null-controllable initial conditions necessarily have a regular hyperbolic component. Indeed, in (10), the source term $A_{12}\partial_x f_2$ entering the hyperbolic equation on f_1 — that has to serve as an indirect control for f_1 — is smooth, because of the parabolic smoothing on f_2. Such a smooth source term cannot steer to zero non-smooth initial conditions.

Theorem 4. — *Let ω be an open interval of $\mathbb{T}$. We assume* (H.1)–(H.4)*, $m=d_2$, $M_1=0$, $M_2=I_{d_2}$, $K_{11}=0$ and $K_{12}=0$. Let T^* be defined by* (3)*. The following statements are equivalent.*

– *For every $T>T^*$ and $f_0=(f_{01},f_{02})\in H^{d_1+1}_{\mathrm{m}}(\mathbb{T})^{d_1}\times H^{d_1+1}(\mathbb{T})^{d_2}$ there exists $u_2\in L^2((0,T)\times\omega)^{d_2}$ such that the solution of* (10) *satisfies $f(T)=0$.*

– *The pair of matrices (A',A_{12}) satisfies the Kalman rank condition:*

$$\operatorname{Span}\{(A')^j A_{12}X_2;\, X_2\in\mathbb{C}^{d_2},\, 0\leqslant j\leqslant d_2-1\}=\mathbb{C}^{d_2}. \tag{12}$$

In Theorem 4, we assume that the open set of control ω is an interval because the proof uses [1, Lem. 2.6] (see Lemma 49 below). The generalization of this result to a general open set ω of $\mathbb{T}$ is not known.

A similar statement can be obtained with the same proof, when $K_{11}=0$, $A_{12}=0$ under Kalman rank condition on (A',K_{12}).

The proof of Theorem 4 follows essentially the same strategy as that of Theorem 3: a strengthened version of Theorem 2 and a cascade structure ensured by Kalman condition. The regularity assumption on the hyperbolic component allows the elimination of the observation of the hyperbolic component.

In time $T<T^*$, we conjecture the existence of

$$f_0=(f_{01},f_{02})\in H^{d_1+1}_{\mathrm{m}}(\mathbb{T})^{d_1}\times H^{d_1+1}(\mathbb{T})^{d_2}$$

such that, for any $u_2\in L^2((0,T)\times\omega)^{d_2}$ the solution of (10) satisfies $f(T)\neq 0$. This fact is not a consequence of Theorem 2 because the functional spaces for f_0 are different, nor a byproduct of the strategy developed in Section 3. It seems this could be proved with holomorphic function technique similar to those used in [19], or with some propagation of singularities, but this is well outside the scope of the article.

After Theorem 4, two problems are still open:

– the characterization of null-controllable initial conditions: it may be a larger space than $H^{d_1+1}_{\mathrm{m}}(\mathbb{T})^{d_1} \times H^{d_1+1}(\mathbb{T})^{d_2}$, see Section 7,

– the algebraic necessary and sufficient condition for null-controllability, involving both matrices A and K. In the context of parabolic systems, this difficulty already appeared, see [4] and [5].

1.3. Organization of the article. — Section 2 is dedicated to preliminary results concerning the spectral analysis of $-B\partial_x^2 + A\partial_x + K$ on $\mathbb{T}$, the well-posedness of (Sys) and the Hilbert uniqueness method.

In Section 3, we prove the negative null-controllability result in time $T < T^*$ of Theorem 2. In Section 4, we prove the positive null-controllability result in time $T > T^*$ of Theorem 2. In Section 5, we explain how to adapt this proof to get the null-controllability in time $T > T^*$ of system (7). The interest of this section is to introduce the proof strategy of Theorem 3 and Theorem 4, in a less technical framework. Then, in Section 6, we prove Theorem 3 and in Section 7, we prove Theorem 4.

1.4. Bibliographical comments

1.4.1. *Wave equation with structural damping.* — We consider the 1D wave equation with structural damping and control h

$$\partial_t^2 y - \partial_x^2 y - \partial_t\partial_x^2 y + b\partial_t y = h(t,x), \tag{13}$$

where $b \in \mathbb{R}$. This equation can be split in a system of the form (Sys) by considering $z := \partial_t y - \partial_x^2 y + (b-1)y$,

$$\begin{cases} \partial_t z + z + (1-b)y = h(t,x), \\ \partial_t y - \partial_x^2 y - z + (b-1)y = 0, \end{cases} \tag{14}$$

i.e., (Sys) with $d=2$, $d_1 = d_2 = 1$, $m=1$,

$$f = \begin{pmatrix} z \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 1-b \\ -1 & b-1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{15}$$

Rosier and Rouchon [26] studied the equation (13) on a 1D-interval, $x \in (0,1)$, with a boundary control at $x = 1$ and $h = 0$. This is essentially equivalent to take (13) with $x \in (0,1)$, Dirichlet boundary conditions at $x = 0$ and $x = 1$, and a source term of the form $h(t,x) = u(t)p(x)$, where p is a fixed profile and u is a scalar control. The authors prove that this equation is not controllable.

By Theorem 2, we extend this negative result to general controls h (i.e., without separate variables) for periodic boundary conditions. Here, $A' = 0$, $\mu_* = 0$, $T^* = +\infty$, the system (14) is not controllable even with an additional control in the second equation.

In [26], the authors prove that this system is not even spectrally controllable, because of an accumulation point in the spectrum. Indeed, by the moment method, a control that would steer the system from an eigenstate to another one would have a Fourier transform vanishing on a set with an accumulation point, which is not possible for an holomorphic function.

Martin, Rosier and Rouchon [23], studied the null-controllability of the equation (13) on the 1D torus, $x \in \mathbb{T}$, with moving controls, i.e., $h(t,x) = u(t,x)1_{\omega+ct}$ with $c \in \mathbb{R}^*$. By the change of variable $x \leftarrow\!\shortmid (x-ct)$, this is equivalent to study the null-controllability of the system

$$\begin{cases} \partial_t z - c\partial_x z + z + (1-b)y = u(t,x)1_\omega(x), \\ \partial_t y - c\partial_x y - \partial_x^2 y - z + (b-1)y = 0 \end{cases} \tag{16}$$

which has the form (Sys) with the same matrices f, B, K as in (15) and

$$A = \begin{pmatrix} -c & 0 \\ 0 & -c \end{pmatrix}.$$

In [23, Th. 1.2], for $c=1$, the authors prove that any initial data $(y_0, y_1) \in H^{s+2} \times H^s(\mathbb{T})$ with $s > 15/2$ can be steered to 0 in time $T > 2\pi$ by means of a control $u \in L^2((0,T)\times\omega)$.

By Theorem 3, we recover this positive null-controllability result with a smaller minimal time $T > \ell(\omega)/|c|$ and a weaker regularity assumption on the initial data $(y, \partial_t y)(0) = (y_0, y_1) \in H^2 \times L^2(\mathbb{T})$ for (13). This corresponds to an initial data $(y,z)(0) \in L^2(\mathbb{T})^2$ for (16) because $z(0) = y_1 - \partial_x^2 y_0 + (b-1)y_0$. Actually, Theorem 3 can be applied for $b = 1$ in (16) but an easy adaptation of Theorem 3 gives the same result for every $b \in \mathbb{R}$. We also prove the negative result in time $T < \ell(\omega)/|c|$. Here, $\mu_* = |c|$, $A_{21} = 0$ and $K_{21} = -1$.

The limitations in [23, Th. 1.2] (regularity and time) are due to the use of controls with separate variables $u(t,x) = u_1(t)u_2(x)$. The proof relies on the moment method and the construction of a biorthogonal family. A key point in both [23] and the present article is a splitting of the spectrum in one parabolic-type part, and one hyperbolic-type part.

Finally, Chaves-Silva, Rosier and Zuazua [9] study the multi-dimensional case of Equation (13), $x \in \Omega$, with Dirichlet boundary conditions and locally distributed moving controls $h(t,x) = u(t,x)1_{\omega(t)}(x)$. The control region $\omega(t)$ is assumed to be driven by the flow of an ODE that covers all the domain Ω within the allotted time T. Then, the authors prove the null-controllability of any initial data $(y_0, y_1) \in H^2 \cap H_0^1(\Omega) \times L^2(\Omega)$ with a L^2-control.

In the particular case $\Omega = \mathbb{T}$ with a motion with constant velocity, Theorem 3 gives the same minimal time for the null-controllability and also the negative result in smaller time.

The proof strategy in [9] consists in proving Carleman estimates for the parabolic equation and the ODE in (14) with *the same singular weight*, adapted to the geometry of the moving support of the control.

As explained in [9, §5.2], the same construction cannot be used with periodic boundary conditions.

In the very recent preprint [16], the authors propose another construction of a weight, to get Carleman estimates for parabolic and transport equations in the torus $\mathbb{T}^2$ (with the same weight). In the present article, we develop another strategy.

1.4.2. *Wave-parabolic systems.* — Albano and Tataru [2] consider 2×2 parabolic-wave systems with boundary control, where

– the coupling term in the wave equation is given by a second order operator with respect to x,

– the coupling term in the parabolic equation is given by a first order operator with respect to (t,x).

This large class contains the linear system of thermoelasticity

$$\begin{cases} \partial_t^2 w - \Delta w + \alpha\Delta\theta = 0, & (t,x)\in(0,T)\times\Omega,\\ \partial_t\theta - \nu\Delta\theta + \beta\partial_t w = 0, & (t,x)\in(0,T)\times\Omega,\\ w(t,x) = u_1(t,x), & (t,x)\in(0,T)\times\partial\Omega,\\ \theta(t,x) = u_2(t,x), & (t,x)\in(0,T)\times\partial\Omega, \end{cases} \tag{17}$$

where $\alpha,\beta,\nu>0$.

The authors of [2] prove the null-controllability in large time of these systems, precisely in any time $T > 2\sup\{|x|; x\in\Omega\}$ for the system (17). The proof relies on Carleman estimates for the heat and the wave equation with the same singular weight. This strategy inspired Chaves-Silva, Rosier and Zuazua [9].

Lebeau and Zuazua [22] prove the null-controllability of the linear system of thermoelasticity (17) with a locally distributed control in the source term of the wave equation, under the geometric control condition on (Ω,ω,T). The method is based on a spectral decomposition. For high frequencies, the spectrum splits into a parabolic part and a hyperbolic part. Projecting the dynamics onto the parabolic/hyperbolic subspaces, the system is decomposed into two weakly coupled systems, the first one behaving like a wave equation, the second one like a heat equation. The wave equation is handled by using the microlocal techniques developed for the wave equation [6]. The parabolic equation is treated by using Lebeau and Robbiano's method [21]. The low frequency part is treated by a compactness argument relying on a unique continuation property.

The proof of the positive controllability results in the present article is an adaptation, to coupled transport-parabolic systems *of any size*, of this approach, introduced for a 2×2 wave-parabolic system. The transport equation is handled by using the results of Alabau-Boussouira, Coron and Olive [1].

The framework of systems (Sys) does not cover the system (17) because the order of the coupling terms is too high.

1.4.3. *Heat equation with memory.* — Ivanov and Pandolfi [17] and after them Guerrero and Imanuvilov [15] consider the heat equation with memory

$$\begin{cases} \partial_t y - \Delta y - \displaystyle\int_0^t \Delta y(\tau)\,\mathrm{d}\tau = u1_\omega, & (t,x)\in(0,T)\times\Omega,\\ y(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega. \end{cases} \tag{18}$$

In 1D, this equation can be split into a system of the form (Sys) by considering $v(t,x) = -\int_0^t y_x(\tau)\,\mathrm{d}\tau$:

$$
\begin{cases}
\partial_t v + \partial_x y = 0,\\
\partial_t y - \partial_x^2 y + \partial_x v = h 1_\omega,\\
y(t,0) = y(t,1) = 0,
\end{cases}
\tag{19}
$$

i.e.,

$$
f = \begin{pmatrix} v\\ y\end{pmatrix},\quad B = \begin{pmatrix} 0\ 0\\ 0\ 1\end{pmatrix},\quad A = \begin{pmatrix} 0\ 1\\ 1\ 0\end{pmatrix},\quad K = \begin{pmatrix} 0\ 0\\ 0\ 0\end{pmatrix}.
$$

In [17], the authors prove that the heat equation with memory term is not "null-controllable to the rest". In [15], the authors prove that the scalar equation (18) is not null-controllable (whatever $T > 0$). Thus the system (19) is not null-controllable.

Theorem 2 proves that, when Dirichlet boundary conditions are replaced by periodic boundary conditions, then system (19) is not null-controllable, even with an additional control in the first equation.

1.4.4. *1D-Linearized compressible Navier-Stokes equations*

The compressible Navier-Stokes equation on the 1D torus writes

$$
\begin{cases}
\partial_t \rho + \partial_x(\rho v) = u_1(t,x) 1_\omega & \text{in } (0,T)\times\mathbb{T}\\
\rho[\partial_t v + v\partial_x v] + \partial_x(a\rho^\gamma) - \mu\partial_x^2 v = u_2(t,x)1_\omega(x) & \text{in } (0,T)\times\mathbb{T}
\end{cases}
$$

where $a,\gamma,\mu > 0$, ρ, v are the density and velocity of the fluid. The state is (ρ, v) and the control is (u_1,u_2). We consider a constant stationary state $(\overline{\rho},\overline{v}) \in \mathbb{R}_+^* \times \mathbb{R}^*$. The linearized system around the trajectory $((\rho,v) = (\overline{\rho},\overline{v}), (u_1,u_2) = (0,0))$ is

$$
\begin{cases}
\partial_t \rho + \overline{v}\partial_x \rho + \overline{\rho}\partial_x v = u_1(t,x)1_\omega, & \text{in } (0,T)\times\mathbb{T}\\
\partial_t v + \overline{v}\partial_x v + a\overline{\rho}^{\gamma-2}\partial_x\rho - \frac{\mu}{\overline{\rho}}\partial_x^2 v = u_2(t,x)1_\omega(x)\,, & \text{in } (0,T)\times\mathbb{T}\,.
\end{cases}
\tag{20}
$$

This system is in the form (Sys) with

$$
f = \begin{pmatrix} \rho\\ v\end{pmatrix},\quad B = \begin{pmatrix} 0 & 0\\ 0 & \mu/\overline{\rho}\end{pmatrix},\quad A = \begin{pmatrix} \overline{v} & \overline{\rho}\\ a\overline{\rho}^{\gamma-2} & \overline{v}\end{pmatrix},\quad K = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}
$$

and satisfies (H.1)–(H.4).

By Theorem 2, the system (20) with two controls (u_1,u_2) is null-controllable in time $T > \ell(\omega)/|\overline{v}|$ and is not null-controllable in time $T < \ell(\omega)/|\overline{v}|$.

By Theorem 3, the system (20) with one control u_1 in the first line (i.e., $u_2 = 0$) is null-controllable in time $T > \ell(\omega)/|\overline{v}|$.

By Theorem 4, with one control u_2 in the second equation (i.e., $u_1 = 0$), any initial condition $(\rho_0, v_0) \in H^2_{\mathrm{m}}(\mathbb{T}) \times H^2(\mathbb{T})$ can be steered to zero in time $T > \ell(\omega)/|\overline{v}|$ by a control $u_2 \in L^2((0,T)\times\omega)$.

In [14], Ervedoza, Glass, Guerrero and Puel consider the (nonlinear) compressible Navier-Stokes equations on a bounded interval $x \in (0,L)$, without source term (i.e., $u_1 = u_2 = 0$), but with a boundary control on both ρ and v at the two boundaries $x = 0$ and $x = L$. They prove the local controllability of this nonlinear system, around the trajectory $(\rho, v) = (\overline{\rho}, \overline{v})$, in appropriate functional spaces. A key ingredient is the controllability of the linearized system, which is proved to hold in time $T > L/|\overline{v}|$. Theorem 2 of the present article enables us to recover the same result with interior control instead of boundary control, and also proves the negative result in small time.

In [11, Th. 1.4], Chowdhury, Mitra, Ramaswamy and Renardy prove the null-controllability of (20) with two controls (u_1, u_2) in time $T > 2\pi/|\overline{v}|$, with spectral methods. Thus, Theorem 2 of the present article enables us to recover the same result but with a better minimal time $\ell(\omega)/|\overline{v}|$ and also proves the negative result in time $T < \ell(\omega)/|\overline{v}|$.

In [10, Th. 1.3], Chowdhury and Mitra prove with moment methods that any initial condition $(\rho_0, v_0) \in H^{s+1}_{\mathrm{m}}(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 6.5$ can be steered to zero in time $T > 2\pi/\overline{v}$ by a control acting only on the second equation $u_2 \in L^2((0,T)\times\omega)$ (i.e., $u_1 = 0$). In [11, Th. 1.2], Chowdhury, Mitra, Ramaswamy and Renardy prove the same result for any initial conditions $(\rho_0, v_0) \in H^1_{\mathrm{m}}(\mathbb{T}) \times L^2(\mathbb{T})$. Thus, Theorem 4 of the present article provides a smaller minimal time $\ell(\omega)/|\overline{v}|$, for smoother initial conditions $(\rho_0, v_0) \in H^2_{\mathrm{m}}(\mathbb{T}) \times H^2(\mathbb{T})$. It also proves the negative result in time $T < \ell(\omega)/|\overline{v}|$.

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2. Preliminary results

We want to understand the operator

$$\mathscr{L} := -B\partial_x^2 + A\partial_x + K \tag{21}$$

with domain

$$D(\mathscr{L}) = \left\{f \in L^2(\mathbb{T})^d; -B\partial_x^2 f + A\partial_x f + Kf \in L^2(\mathbb{T})^d\right\} \tag{22}$$

where the derivatives are considered in the distributional sense $\mathscr{D}'(\mathbb{T})$. Throughout the article, we will note e_n the function $x \mapsto \mathrm{e}^{inx}$. We remark that applying $\mathscr{L}$ to Xe_n, where $X \in \mathbb{C}^d$, we get

$$\mathscr{L}(Xe_n) = n^2\Big(B + \frac{\mathrm{i}}{n}\,A + \frac{1}{n^2}\,K\Big)Xe_n. \tag{23}$$

Thus, if we define the following perturbation $E(z)$ of B

$$\forall\, z \in \mathbb{C},\ E(z) = B + zA - z^2K, \tag{24}$$

then $\mathscr{L}$ acts on the Fourier side as multiplication by $n^2E(\mathrm{i}/n)$.

In Section 2.1, we apply perturbation theory to the matrices $E(z)$ near $z=0$: the spectrum of $E(z)$ splits into two parts: one close to zero that defines the hyperbolic component, one close to the spectrum of D that defines the parabolic component. In Section 2.2, we deduce the dissipation of the parabolic component and the boundedness of the hyperbolic component. Thanks to these estimates, we prove the well-posedness of the system (Sys). Finally, in Section 2.3, we recall the Hilbert Uniqueness Method.

2.1. Perturbation theory. — If we want to understand the semigroup $e^{t\mathscr{L}}$, we need to know the spectrum and the eigenvectors of $E(z)$. Here, we relate the spectral properties of $E(z)$ to those of A and B, in the limit $z \to 0$. This is instrumental in all the article. Our proofs are essentially self-contained, but the reader unfamiliar with analytic perturbation theory in finite dimension may read [18, Ch. II, §1 & §2].

For $r > 0$ and $m \in \mathbb{N}^*$, we define $\mathscr{O}_r^{m\times m}$ as the set of holomorphic functions in the complex disk $D(0,r)$ with values in $\mathbb{C}^{m\times m}$. Our first result is the following one.

Proposition 5. — *There exist $r > 0$ and a matrix-valued holomorphic function $P^{\mathrm{h}} \in \mathscr{O}_r^{d\times d}$ such that*

(i) $P^{\mathrm{h}}(0) = \left(\begin{smallmatrix} I_{d_1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$,

(ii) *for all $|z| < r$, $P^{\mathrm{h}}(z)$ is a projection that commutes with $E(z)$,*

(iii) *in the limit $z \to 0$, $E(z)P^{\mathrm{h}}(z) = O(z)$.*

Proof. — The spectrum of $E(z)$ is continuous in z (see [18, Ch. II, §1.2]). Let us consider the "0-group" of eigenvalues, i.e., the set of eigenvalues that tend to 0 as $z \to 0$. Then we note $P^{\mathrm{h}}(z)$ the sum of the projections onto the eigenspace[(3)] of $E(z)$ associated with eigenvalues in the 0-group along the other eigenspaces. Another way to define $P^{\mathrm{h}}(z)$ is to choose $R = \frac{1}{2}\min_{\lambda\in\mathrm{Sp}(D)} |\lambda|$ and r small enough so that for $|z| < r$, there is no eigenvalues of $E(z)$ on the circle $\partial D(0,R)$. Then, we define (see [18, Ch. II, Eq. (1.16)])

$$P^{\mathrm{h}}(z) = -\frac{1}{2\mathrm{i}\pi}\int_{\partial D(0,R)} (E(z) - \zeta I_d)^{-1}\,\mathrm{d}\zeta. \tag{25}$$

In the terminology of Kato, $P^{\mathrm{h}}(z)$ is the "total projection for the 0-group". Then, according to [18, Ch. II, §1.4], $P^{\mathrm{h}}(z)$ is the projection onto the sum of eigenspaces associated to eigenvalues of $E(z)$ lying inside $D(0,R)$ along the other eigenspaces. It is holomorphic in $|z| < r$. For $z = 0$, the formula (25) that defines $P^{\mathrm{h}}(0)$ becomes

$$P^{\mathrm{h}}(0) = -\frac{1}{2\mathrm{i}\pi}\int_{\partial D(0,R)} (B - \zeta I_d)^{-1}\,\mathrm{d}\zeta.$$

Then, $P^{\mathrm{h}}(0)$ is the projection onto the eigenspace of B associated to the eigenvalue 0 along the other eigenspaces (see [18, Ch. II, §1.4]). So, according to the hypotheses (H.2)–(H.3) on the blocks of B, $P^{\mathrm{h}}(0) = \left(\begin{smallmatrix} I_{d_1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This proves (i).

According to the definition (25), $P^{\mathrm{h}}(z)$ commutes with $E(z)$. This proves (ii). Then we have

$$P^{\mathrm{h}}(0)E(0) = E(0)P^{\mathrm{h}}(0) = BP^{\mathrm{h}}(0) = 0,$$

which, along with the holomorphy of P^{h}, proves (iii). □

We say that P^{h} is the "projection on the hyperbolic branches". We define $P^{\mathrm{p}}(z) = I_d - P^{\mathrm{h}}(z)$, which we call the "projection on the parabolic branches", and satisfies properties analog to P^{h}:

[(3)] We stress that when we talk about "eigenspace", we mean "generalized eigenspace" (or, in the terminology of Kato, algebraic eigenspace), i.e., the space of generalized eigenvectors.

Proposition 6. — *The matrix-valued function P^{p} is in $\mathscr{O}_r^{d\times d}$ and*

(i) $P^{\mathrm{p}}(0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{d_2}\end{smallmatrix}\right)$,

(ii) *for all $|z| < r$, $P^{\mathrm{p}}(z)$ is a projection that commutes with $E(z)$,*

(iii) *in the limit $z \to 0$, $E(z)P^{\mathrm{p}}(z) = B + O(z)$.*

We will need to split the hyperbolic branches further. Let us recall that A' is the matrix multiplying the derivative of the hyperbolic components in the system (Sys) (see Equation (1)), i.e., we have $A = \left(\begin{smallmatrix} A' & A_{12} \\ A_{21} & A_{22}\end{smallmatrix}\right)$.

Proposition 7. — *There exist $r > 0$ and a family of matrix-valued holomorphic functions $(P^{\mathrm{h}}_\mu)_{\mu\in\mathrm{Sp}(A')} \in (\mathscr{O}_r^{d\times d})^{\mathrm{Sp}(A')}$ satisfying*

(i) *for all $\mu \in \mathrm{Sp}(A')$ and $|z| < r$, $P^{\mathrm{h}}_\mu(z)$ is a non-zero projection that commutes with $E(z)$,*

(ii) *for all $|z| < r$, $P^{\mathrm{h}}(z) = \sum_{\mu\in\mathrm{Sp}(A')} P^{\mathrm{h}}_\mu(z)$ and for all $\mu \neq \mu'$, $P^{\mathrm{h}}_\mu(z)P^{\mathrm{h}}_{\mu'}(z) = 0$,*

(iii) *for every $\mu \in \mathrm{Sp}(A')$, there exists $R^{\mathrm{h}}_\mu \in \mathscr{O}_r^{d\times d}$ such that*

$$\forall\, |z| < r,\ E(z)P^{\mathrm{h}}_\mu(z) = \mu z P^{\mathrm{h}}_\mu(z) + z^2 R^{\mathrm{h}}_\mu(z).$$

Remark 8. — For $\mu \in \mathrm{Sp}(A')$, the projection P^{h}_μ is holomorphic and thus continuous in $D(0,r)$. Therefore, the rank of $P^{\mathrm{h}}_\mu(z)$, which is its trace, does not depend on $|z| < r$ (the $P^{\mathrm{h}}_\mu(z)$ even are similar, see [18, Ch. I, §4.6, Lem. 4.10]). In the same vein, the ranks of $P^{\mathrm{h}}(z)$ and $P^{\mathrm{p}}(z)$ do not depend on z.

Proof. — The proof is essentially the "reduction process" of Kato [18, Ch. II, §2.3]. According to Proposition 5, P^{h} is holomorphic and $P^{\mathrm{h}}(z)E(z) = O(z)$. Then we define

$$E^{(1)}(z) = z^{-1}E(z)P^{\mathrm{h}}(z) = z^{-1}P^{\mathrm{h}}(z)E(z),$$

which is holomorphic in $|z| < r$. Note that we have according to Kato [18, Ch. II, Eq. (2.38)]

$$E^{(1)}(0) = P^{\mathrm{h}}(0)E'(0)P^{\mathrm{h}}(0) = \begin{pmatrix} A' & 0 \\ 0 & 0\end{pmatrix}.$$

Let us assume for the moment that 0 is not an eigenvalue of A'. Then, for $\mu \in \mathrm{Sp}(A')$, we define $P^{\mathrm{h}}_\mu(z)$ the total projection on the μ-group of eigenvalues of $E^{(1)}(z)$. Said otherwise, and according to the definition of $E^{(1)}(z)$, $P^{\mathrm{h}}_\mu(z)$ is the total projection on the μz-group of eigenvalues of $E(z)$. The projection $P^{\mathrm{h}}_\mu(z)$ is defined and holomorphic for z small enough according to [18, Ch. II, §1.4].

Since for z small enough, $P^{\mathrm{h}}_\mu(z)$ is the projection on some eigenspaces of $E^{(1)}(z)$ associated with non-zero eigenvalues,

$$\mathrm{Im}(P^{\mathrm{h}}_\mu(z)) \subset \mathrm{Im}(E^{(1)}(z)) \subset \mathrm{Im}(P^{\mathrm{h}}(z)),$$

with the last inclusion coming from the definition of $E^{(1)}(z)$. Thus $P^{\mathrm{h}}_\mu(z)$ is a subprojection of $P^{\mathrm{h}}(z)$. Moreover, $P^{\mathrm{h}}_\mu(z)$ commutes with $E^{(1)}(z)$, so it commutes with $E(z)$. This proves Item (i) in the case $0 \notin \mathrm{Sp}(A')$.

For $\mu \neq \nu$, $P^{\mathrm{h}}_\mu(z)$ and $P^{\mathrm{h}}_\nu(z)$ are the projections on some sums of eigenspaces associated with different eigenvalues, so $P^{\mathrm{h}}_\mu(z)P^{\mathrm{h}}_\nu(z) = 0$. Let us denote for convenience $Q^{\mathrm{h}}(z) = \sum_{\mu\in \mathrm{Sp}(A')} P^{\mathrm{h}}_\mu(z)$. Then, for z small, $Q^{\mathrm{h}}(z)$ is the projection on all the eigenspaces of $E^{(1)}(z)$ associated with non-zero eigenvalues. According to the definition of $E^{(1)}(z)$, this proves that $Q^{\mathrm{h}}(z)$ is a subprojection of $P^{\mathrm{h}}(z)$. Let us check that $Q^{\mathrm{h}}(z)$ and $P^{\mathrm{h}}(z)$ have the same rank. This will prove that for all z small enough, $Q^{\mathrm{h}}(z) = P^{\mathrm{h}}(z)$. The rank of $Q^{\mathrm{h}}(z)$, which is its trace, does not depend on z. The same is true for $P^{\mathrm{h}}(z)$. For $z = 0$, we have $E^{(1)}(0) = \left(\begin{smallmatrix} A' & 0 \\ 0 & 0\end{smallmatrix}\right)$, so by using the fact that $0 \notin \mathrm{Sp}(A')$,

$$Q^{\mathrm{h}}(0) = \begin{pmatrix} I_{d_1} & 0 \\ 0 & 0\end{pmatrix} = P^{\mathrm{h}}(0).$$

This proves that for all z small enough, $Q^{\mathrm{h}}(z) = P^{\mathrm{h}}(z)$, and in turn finishes the proof of Item (ii) in the case where $0 \notin \mathrm{Sp}(A')$.

If $0 \in \mathrm{Sp}(A')$, then we add $\alpha z I$ to $E(z)$ for some $\alpha \in \mathbb{C}$. This amounts to adding $\alpha P^{\mathrm{h}}(z)$ to $E^{(1)}(z)$. This only shifts the eigenvalues of the restriction of $E^{(1)}(z)$ to $\mathrm{Im}(P^{\mathrm{h}}(z))$ (but not of its restriction to $\mathrm{Im}(I_d - P^{\mathrm{h}}(z))$) by α, while leaving the eigenprojections unchanged. Thus, choosing α so that $0 \notin \alpha + \mathrm{Sp}(A')$, we get Items (i) and (ii) in the case $0 \in \mathrm{Sp}(A')$.

We still need to prove the asymptotics of Item (iii). Since A' is diagonalizable, so is $E^{(1)}(0) = \left(\begin{smallmatrix} A' & 0 \\ 0 & 0\end{smallmatrix}\right)$. So, there is no nilpotent part in the spectral decomposition of $E^{(1)}(0)$. That is to say, for all $\mu \in \mathrm{Sp}(A')$,

$$E^{(1)}(0)P^{\mathrm{h}}_\mu(0) = \mu P^{\mathrm{h}}_\mu(0).$$

Since $z \mapsto E^{(1)}(z)P^{\mathrm{h}}_\mu(z)$ is holomorphic, we have

$$E^{(1)}(z)P^{\mathrm{h}}_\mu(z) = \mu P^{\mathrm{h}}_\mu(z) + O(z).$$

Finally, we multiply by z to come back to $E(z)$, which gives us

$$E(z)P^{\mathrm{h}}_\mu(z) = \mu z P^{\mathrm{h}}_\mu(z) + O(z^2). \qquad \square$$

2.2. Estimates on Fourier components and well-posedness

2.2.1. *Dissipation of the parabolic component*. — The goal of this section is the proof of the following result.

Proposition 9. — *There exist $r, K_{\mathrm{p}}, c_{\mathrm{p}} > 0$ such that for every $|z| < r$, $\tau > 0$ and $X \in \mathrm{Im}(P^{\mathrm{p}}(z))$,*

$$|\mathrm{e}^{-E(z)\tau}X| \leqslant K_{\mathrm{p}}\mathrm{e}^{-c_{\mathrm{p}}\tau}|X|.$$

Proof. — By using Proposition 6, for $|z| \leqslant r$, we denote by $E^{\mathrm{p}}(z)$ the restriction of $E(z)$ to the vector subspace $\mathrm{Im}[P^{\mathrm{p}}(z)]$, which is an endomorphism of $\mathrm{Im}[P^{\mathrm{p}}(z)]$.

By assumption (H.3), there exists $c > 0$ such that $\Re(\mathrm{Sp}(D)) \subset (c, \infty)$. There exists an open disk Ω in the complex plane such that $\mathrm{Sp}(D) \subset \Omega$ and $\min\{\Re(z);\ z \in \overline{\Omega}\} > c$. Then, by continuity of the spectrum, for r small enough, we have, for every $|z| \leqslant r$, $\mathrm{Sp}(E^{\mathrm{p}}(z)) \subset \Omega$.

Step 1: Cauchy formula. — We prove the following equality between endomorphisms of $\mathrm{Im}[P^{\mathrm{p}}(z)]$

$$\forall\,|z|\leqslant r,\ \tau\in\mathbb{R},\quad \mathrm{e}^{-E^{\mathrm{p}}(z)\tau}=\frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega}\mathrm{e}^{-\tau\xi}(\xi I-E^{\mathrm{p}}(z))^{-1}\,\mathrm{d}\xi, \tag{26}$$

where I is the identity on $\mathrm{Im}[P^{\mathrm{p}}(z)]$. The right-hand side is well-defined because $\partial\Omega\cap\mathrm{Sp}(E^{\mathrm{p}}(z))=\varnothing$. Let us denote it by $\phi(\tau)$. Then

$$\begin{aligned}\phi'(\tau)&=\frac{-1}{2\pi\mathrm{i}}\int_{\partial\Omega}\mathrm{e}^{-\tau\xi}\xi(\xi I-E^{\mathrm{p}}(z))^{-1}\,\mathrm{d}\xi\\&=\frac{-1}{2\pi\mathrm{i}}\int_{\partial\Omega}\mathrm{e}^{-\tau\xi}((\xi I-E^{\mathrm{p}}(z))+E^{\mathrm{p}}(z))(\xi I-E^{\mathrm{p}}(z))^{-1}\,\mathrm{d}\xi.\end{aligned}$$

By the Cauchy formula, $\int_{\partial\Omega}\mathrm{e}^{-\tau\xi}\,\mathrm{d}\xi=0$ thus $\phi'(\tau)=-E^{\mathrm{p}}(z)\phi(\tau)$. Moreover $\phi(0)=I$ because all the eigenvalues of $E^{\mathrm{p}}(z)$ are inside Ω (see [18, Ch. I, Prob. 5.9]). Thus $\phi(\tau)=\mathrm{e}^{-\tau E^{\mathrm{p}}(z)}$.

Step 2: Estimate. — We deduce from (26) the following equality between endomorphisms of $\mathbb{C}^d$

$$\forall\,|z|\leqslant r,\ \tau\in\mathbb{R},\quad \mathrm{e}^{-E(z)\tau}P^{\mathrm{p}}(z)=\frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega}\mathrm{e}^{-\tau\xi}(\xi I_d-E(z))^{-1}P^{\mathrm{p}}(z)\,\mathrm{d}\xi. \tag{27}$$

Note that, if r is small enough, then the eigenvalues of $E(z)$ are either inside Ω (parabolic branch) or close to 0 (hyperbolic branch), for instance in $\{\Re(\xi)<c/2\}$. Thus $(\xi I_d-E(z))$ is invertible on $\mathbb{C}^d$ for every $\xi\in\partial\Omega$ and the above right-hand side is well-defined.

We deduce from (27) that

$$\big|\mathrm{e}^{-E(z)\tau}P^{\mathrm{p}}(z)\big|\leqslant\frac{1}{2\pi}\int_{\partial\Omega}\mathrm{e}^{-\tau\Re(\xi)}\big|(\xi I_d-E(z))^{-1}P^{\mathrm{p}}(z)\big|\,\mathrm{d}\xi.$$

The map $(\xi,z)\in\partial\Omega\times\overline{D}(0,r)\mapsto\big|(\xi I_d-E(z))^{-1}P^{\mathrm{p}}(z)\big|$ is continuous on a compact set thus bounded. Thus there exists a positive constant K such that, for every $|z|<r$ and $\tau>0$, $\big|\mathrm{e}^{-E(z)\tau}P^{\mathrm{p}}(z)\big|\leqslant K\mathrm{e}^{-c\tau}$. □

2.2.2. *Boundedness of the transport component*. — The goal of this section is to prove the following result.

Proposition 10. — *There exist $r,K_{\mathrm{h}},c_{\mathrm{h}}>0$ such that for every $x\in[-r,r]\smallsetminus\{0\}$, $t\in\mathbb{R}$ and $X\in\mathrm{Im}(P^{\mathrm{h}}(\mathrm{i}x))$,*

$$\Big|\exp\Big(\frac{t}{x^2}E(\mathrm{i}x)\Big)X\Big|\leqslant K_{\mathrm{h}}\,\mathrm{e}^{c_{\mathrm{h}}|t|}|X|.$$

Proof. — Let r be as in Proposition 7, $x\in[-r,r]\smallsetminus\{0\}$, $t\in\mathbb{R}$, $\mu\in\mathrm{Sp}(A')$ and $Y\in\mathrm{Im}[P^{\mathrm{h}}_\mu(\mathrm{i}x)]$. Taking into account that $\mathrm{Im}[P^{\mathrm{h}}_\mu(\mathrm{i}x)]$ is stable by $E(\mathrm{i}x)$, we get

$$\exp\Big(\frac{t}{x^2}E(\mathrm{i}x)\Big)Y=\exp\Big(\frac{t}{x^2}E(\mathrm{i}x)P^{\mathrm{h}}_\mu(\mathrm{i}x)\Big)Y=\exp\Big(\frac{t}{x^2}\big(\mathrm{i}\mu xP^{\mathrm{h}}_\mu(\mathrm{i}x)-x^2R^{\mathrm{h}}_\mu(\mathrm{i}x)\big)\Big)Y.$$

Note that $P_\mu^{\mathrm{h}}(\mathrm{i}x)$ and $R_\mu^{\mathrm{h}}(\mathrm{i}x)$ commute because $P_\mu^{\mathrm{h}}(\mathrm{i}x)$ and $E(\mathrm{i}x)$ commute and $E(\mathrm{i}x)P_\mu^{\mathrm{h}}(\mathrm{i}x) = \mu\mathrm{i}xP_\mu^{\mathrm{h}}(\mathrm{i}x) - x^2R_\mu^{\mathrm{h}}(\mathrm{i}x)$. Thus, by using that $\mathrm{i}\mu/x \in \mathrm{i}\mathbb{R}$, we obtain

$$\Big|\exp\Big(\frac{t}{x^2}E(\mathrm{i}x)\Big)Y\Big| = \Big|\mathrm{e}^{\mathrm{i}\mu t/x}\exp\big(-tR_\mu^{\mathrm{h}}(\mathrm{i}x)\big)Y\Big| \leqslant \mathrm{e}^{c_\mu|t|}|Y|,$$

where $c_\mu = \max\{|R_\mu^{\mathrm{h}}(z)|;\ z \in \overline{D}(0,r)\}$. We conclude for $X \in \mathrm{Im}[P^{\mathrm{h}}(\mathrm{i}x)]$ that

$$\begin{aligned}\Big|\exp\Big(\frac{t}{x^2}E(\mathrm{i}x)\Big)X\Big| &\leqslant \sum_{\mu\in\mathrm{Sp}(A')}\Big|\exp\Big(\frac{t}{x^2}E(\mathrm{i}x)\Big)P_\mu^{\mathrm{h}}(\mathrm{i}x)X\Big|\\ &\leqslant \sum_{\mu\in\mathrm{Sp}(A')}\mathrm{e}^{c_\mu|t|}|P_\mu^{\mathrm{h}}(\mathrm{i}x)X| \leqslant K\mathrm{e}^{c|t|}|X|\end{aligned}$$

with $c = \max\{c_\mu;\ \mu \in \mathrm{Sp}(A')\}$ and $K = \max\big\{\sum_{\mu\in\mathrm{Sp}(A')}|P_\mu^{\mathrm{h}}(z)|;\ z \in \overline{D}(0,r)\big\}$. □

2.2.3. *Well-posedness*. — By gathering the results of Sections 2.2.1 and 2.2.2, we can prove that the heat-transport system (Sys) is well-posed. We define the Fourier coefficients of $f \in L^2(\mathbb{T})^d$ by

$$\forall\, n \in \mathbb{Z},\ \widehat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\mathrm{e}^{-int}\,\mathrm{d}t \in \mathbb{C}^d.$$

We consider the operator $\mathscr{L}$ defined by (21) and (22). By Bessel-Parseval identity and the fact that $\mathscr{L}(Xe_n) = n^2E(\mathrm{i}/n)Xe_n$,

$$D(\mathscr{L}) = \Big\{f \in L^2(\mathbb{T})^d;\ \textstyle\sum_{n\in\mathbb{Z}}\big|n^2E(\mathrm{i}/n)\widehat{f}(n)\big|^2 < \infty\Big\}. \tag{28}$$

The goal of this section is to prove the following result.

Proposition 11. — *$-\mathscr{L}$ generates a C^0 semi-group of bounded operators on $L^2(\mathbb{T}^d)$.*

This result will ensure well-posedness of (Sys) in the following sense.

Definition 12. — Let $T > 0$, $f_0 \in L^2(\mathbb{T})^d$ and $u \in L^2(Q_T)^d$. The solution of (Sys) is the function $f \in C^0([0,T];L^2(\mathbb{T})^d)$ defined for $t \in [0,T]$ by

$$f(t) = \mathrm{e}^{-t\mathscr{L}}f_0 + \int_0^t \mathrm{e}^{-(t-\tau)\mathscr{L}}M\mathbb{1}_\omega u(\tau)\,\mathrm{d}\tau.$$

Moreover, $f(t)$ satisfies the estimate

$$\forall\, 0 \leqslant t \leqslant T,\ \|f(t)\|_{L^2(\mathbb{T})} \leqslant C\big(\|f_0\|_{L^2(\mathbb{T})} + \|u\|_{L^2([0,T]\times\omega)}\big), \tag{29}$$

where C depends on T but not on f_0 and u. We will also note $S(t, f_0, u) := f(t)$ this solution.

Proof. — We deduce from Propositions 9 and 10 that for every $x \in [-r,r] \setminus \{0\}$, $t>0$ and $X \in \mathbb{C}^d$,

$$\begin{aligned}\Big|\exp\Big(-\frac{t}{x^2}E(\mathrm{i}x)\Big)X\Big| &\leqslant \Big|\exp\Big(-E(\mathrm{i}x)\frac{t}{x^2}\Big)P^{\mathrm{p}}(\mathrm{i}x)X\Big| \\ &\qquad + \Big|\exp\Big(-\frac{t}{x^2}E(\mathrm{i}x)\Big)P^{\mathrm{h}}(\mathrm{i}x)X\Big| \\ &\leqslant K_{\mathrm{p}}\,\mathrm{e}^{-c_{\mathrm{p}}tx^{-2}}|P^{\mathrm{p}}(\mathrm{i}x)X| + K_{\mathrm{h}}\,\mathrm{e}^{c_{\mathrm{h}}t}\big|P^{\mathrm{h}}(\mathrm{i}x)X\big| \\ &\leqslant K\,\mathrm{e}^{c_{\mathrm{h}}t}|X|,\end{aligned}\tag{30}$$

where $K=\max\big\{K_{\mathrm{p}}|P^{\mathrm{p}}(\mathrm{i}x)|+K_{\mathrm{h}}\big|P^{\mathrm{h}}(\mathrm{i}x)\big|;x\in[-r,r]\big\}$.

For $f\in L^2(\mathbb{T})^d$ and $t\in[0,\infty)$ we define

$$S(t)=\sum_{n\in\mathbb{Z}}\mathrm{e}^{-tn^2E(\mathrm{i}/n)}\widehat{f}(n)e_n.$$

By Bessel Parseval equality and (30) with $x=1/n$, $S(t)$ is a bounded operator on $L^2(\mathbb{T})^d$, because the number of $n\in\mathbb{Z}$ such that $1/n\notin[-r,r]$ is finite. The semi-group properties $S(0)=I$ and $S(t+s)=S(t)S(s)$ are clearly satisfied. For $f\in D(\mathscr{L})$, we have, by Bessel Parseval equality

$$\Big\|\Big(\frac{S(t)-I}{t}+\mathscr{L}\Big)f\Big\|^2_{L^2(\mathbb{T})^d}=\sum_{n\in\mathbb{Z}}\Big|\Big(\frac{\mathrm{e}^{-tn^2E(\mathrm{i}/n)}-I_d}{t}-n^2E(\mathrm{i}/n)\Big)\widehat{f}(n)\Big|^2.$$

In the right-hand side, each term of the series converges to zero when $[t\to0]$ and, thanks to (30), is dominated for every $t\in[0,1]$ and $n>1/r$ by

$$\Big|\Big(\int_0^1\mathrm{e}^{-t\theta n^2E(\mathrm{i}/n)}\,\mathrm{d}\theta-I_d\Big)n^2E(\mathrm{i}/n)\widehat{f}(n)\Big|^2\leqslant(K\mathrm{e}^{c_h}+1)^2\,\big|n^2E(\mathrm{i}/n)\widehat{f}(n)\big|^2,$$

which can be summed over $n\in\mathbb{Z}$ because $f\in D(\mathscr{L})$, see (28). By the dominated convergence theorem, the sum of the series converges to zero. ☐

Remark 13. — We can see from this proof that the semi-group $\mathrm{e}^{-t\mathscr{L}}$ is strongly continuous on any $H^s(\mathbb{T})^d$ for any $s\geqslant0$, i.e., we have

$$\|\mathrm{e}^{-t\mathscr{L}}f_0\|_{H^s(\mathbb{T})^d}\leqslant K\mathrm{e}^{c_{\mathrm{h}}t}\|f_0\|_{H^s(\mathbb{T})^d}.$$

2.3. Adjoint system and observability. — The null-controllability of a linear system is equivalent to a dual notion called “observability”. We have the following general, abstract result (see [12, Lem. 2.48]).

Lemma 14. — *Let H_1, H_2 and H_3 be three Hilbert spaces. Let $\Phi_2\colon H_2\to H_1$ and $\Phi_3\colon H_3\to H_1$ be continuous linear maps. Then*

$$\operatorname{Im}(\Phi_2)\subset\operatorname{Im}(\Phi_3)$$

if and only if there exists $C>0$ such that

$$\forall\,h_1\in H_1,\ \|\Phi_2^*h_1\|_{H_2}\leqslant C\|\Phi_3^*h_1\|_{H_3}.$$

From Lemma 14, see [12, Th. 2.44], we deduce the following result.

Proposition 15. — *Given $T>0$, the system* (Sys) *is null-controllable on ω in time T if and only if there exists $C>0$ such that for every $g_0 \in L^2(\mathbb{T};\mathbb{C}^d)$, the solution g to the equation*[(4)]

$$\begin{cases} \partial_t g - B^{\mathrm{tr}}\partial_x^2 g - A^{\mathrm{tr}}\partial_x g + K^{\mathrm{tr}} g = 0 & \text{in } (0,T)\times\mathbb{T}, \\ g(0,\cdot) = g_0 & \text{in } \mathbb{T}. \end{cases} \tag{31}$$

satisfies

$$\|g(T,\cdot)\|^2_{L^2(\mathbb{T};\mathbb{C}^d)} \leqslant C\int_0^T \int_\omega |M^* g(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x. \tag{32}$$

Note that the solutions of the adjoint system (31) are of the form[(5)]

$$g(t,x) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-tn^2 E(\mathrm{i}/n)^*}\widehat{g}_0(n)\mathrm{e}^{\mathrm{i}nx}. \tag{33}$$

Moreover, we have a spectral theory for the adjoint system that is similar to Propositions 5–7. We just have to take the adjoint of each formulas of these Propositions.

Remark 16. — As for the semi-group $\mathrm{e}^{-t\mathscr{L}}$ (see Remark 13), the dual semi-group $\mathrm{e}^{-t\mathscr{L}^*}$ is strongly continuous on any $H^s(\mathbb{T})^d$ for any $s \geqslant 0$, i.e., we have

$$\|\mathrm{e}^{-t\mathscr{L}^*} g_0\|_{H^s(\mathbb{T})^d} \leqslant K'\mathrm{e}^{c't}\|g_0\|_{H^s(\mathbb{T})^d}.$$

3. Obstruction to the null-controllability in small time

The goal of this section is to prove the first point of Theorem 2, by disproving the observability inequality (32) on an appropriate solution of the adjoint system (31). Intuitively, the lack of null-controllability in small time should come from the transport components. So, the idea is to construct approximate transport solutions. Note that in general, there are no non-trivial *exact* transport solutions that are supported on a strict subset of $[0,T]\times\mathbb{T}$ (see the appendix).

Proof of the lack of null-controllability in time $T<T^*$

Step 1: Construction of approximate transport solutions. — Let $\mu \in \mathrm{Sp}(A')$ with minimum absolute value (i.e., $|\mu| = \mu^*$). Let $\chi \in C^\infty(\mathbb{T}) \smallsetminus \{0\}$ be such that the solution $\eta(t,x) = \chi(x+\mu t)$ of the transport equation $(\partial_t - \mu\partial_x)\eta(t,x) = 0$ on $[0,T]\times\mathbb{T}$, with initial condition $\eta(0,\cdot) = \chi$, has its support $\mathrm{supp}(\eta)$ disjoint from $[0,T]\times\omega$. Such a solution exists because $T<T^*$.

To exploit the spectral asymptotics of the previous section, that are valid in the high-frequency limit, we want a high-frequency version of χ. To that end, we consider for $N \in \mathbb{N}^*$ the polynomial $P_N(X) = \prod_{j=-N}^{N}(X-j)$ and $\chi_N = P_N(-\mathrm{i}\partial_x)\chi$.

(4) We denote the transpose of $M \in \mathbb{R}^{d\times d}$ by M^{tr}.

(5) When we write $E(z)^*$, it is to be understood as $(E(z))^*$. We will use the same notation for $P^{\mathrm{h}}_\mu(z)^*$ etc.

Since χ_N is the image of χ by a differential operator, we have $\operatorname{supp}(\chi_N)\subset\operatorname{supp}(\chi)$. If we note $\chi(x)=\sum_{n\in\mathbb{Z}}a_n\mathrm{e}^{inx}$ and $\chi_N(x)=\sum_{n\in\mathbb{Z}}a_n^N\mathrm{e}^{inx}$, we have $a_n^N=P_N(n)a_n$. In particular, for $|n|\leqslant N$, $a_n^N=0$.

In summary, χ_N satisfies the following properties:

– χ_N is non-zero,
– χ_N has no components along frequencies less than N,
– the support of χ_N is a subset of the support of χ.

In particular, the last property implies that the solution η_N of $(\partial_t-\mu\partial_x)\eta_N(t,x)=0$, with initial condition $\eta_N(0,\cdot)=\chi_N$ is such that $\operatorname{supp}(\eta_N)$ is disjoint from $[0,T]\times\omega$.

We adopt the notations of Proposition 7. Let $\varphi_0\in\operatorname{Im}(P^{\mathrm{h}}_\mu(0)^*)\smallsetminus\{0\}$. We define

$$\widetilde{g}_N(t,x)=\sum_{n\in\mathbb{Z}}a_n^N\mathrm{e}^{in(x+\mu t)+tR^{\mathrm{h}}_\mu(0)^*}\varphi_0=\chi_N(x+\mu t)\mathrm{e}^{tR^{\mathrm{h}}_\mu(0)^*}\varphi_0,\tag{34}$$

$$g_N(t,x)=\sum_{n\in\mathbb{Z}}a_n^N\mathrm{e}^{in(x+\mu t)+tR^{\mathrm{h}}_\mu(\mathrm{i}/n)^*}P^{\mathrm{h}}_\mu(\mathrm{i}/n)^*\varphi_0.\tag{35}$$

By Proposition 7, $E(z)^*$ acts as $\mu\overline{z}+\overline{z}^2R^{\mathrm{h}}_\mu(z)^*$ on the range of $P^{\mathrm{h}}_\mu(z)^*$. So the definition of g_N can be written alternatively as

$$g_N(t,x)=\sum_{n\in\mathbb{Z}}a_n^N\mathrm{e}^{inx-tn^2E(\mathrm{i}/n)^*}P^{\mathrm{h}}_\mu(\mathrm{i}/n)^*\varphi_0.$$

So, according to the representation of the solutions of the adjoint system (33), g_N solves the parabolic-transport system (31). On the other hand, $\widetilde{g}_N$ solves the transport equation $(\partial_t-\mu\partial_x-R^{\mathrm{h}}_\mu(0)^*)\widetilde{g}_N=0$. We will prove that in the limit $N\to+\infty$, g_N is an approximation of $\widetilde{g}_N$.

Step 2: Approximation of the exact solution by the transport solution. — According to Parseval's identity, we have for every $t\geqslant 0$

$$\|g_N(t,\cdot)-\widetilde{g}_N(t,\cdot)\|^2_{L^2(\mathbb{T})}=2\pi\sum_{n\in\mathbb{Z}}|a_n^N|^2\left|\left(\mathrm{e}^{tR^{\mathrm{h}}_\mu(0)^*}-\mathrm{e}^{tR^{\mathrm{h}}_\mu(\mathrm{i}/n)^*}P^{\mathrm{h}}_\mu(\mathrm{i}/n)^*\right)\varphi_0\right|^2.$$

Then, according to the holomorphy of $z\mapsto R^{\mathrm{h}}_\mu(z)$ and $z\mapsto P^{\mathrm{h}}_\mu(z)$, and the fact that $P^{\mathrm{h}}_\mu(0)^*\varphi_0=\varphi_0$, we have uniformly with respect to $t\in[0,T]$

$$\|g_N(t,\cdot)-\widetilde{g}_N(t,\cdot)\|^2_{L^2(\mathbb{T})}=2\pi\sum_{n\in\mathbb{Z}}|a_n^N|^2O(1/n^2).$$

Now, reminding that for $|n|\leqslant N$, $a_n^N=0$, we deduce that

$$\|g_N(t,\cdot)-\widetilde{g}_N(t,\cdot)\|^2_{L^2(\mathbb{T})}=O\big(1/N^2\big)\sum_{n\in\mathbb{Z}}|a_n^N|^2.$$

Thanks to Parseval's identity we rewrite it as

$$\|g_N(t,\cdot)-\widetilde{g}_N(t,\cdot)\|_{L^2(\mathbb{T})}=O(1/N)\|\chi_N\|_{L^2(\mathbb{T})}.\tag{36}$$

Step 3: Conclusion. — By the triangle inequality, we have for $0 \leqslant t \leqslant T$

$$\|g_N(t,\cdot)\|_{L^2(\omega)} \leqslant \|\widetilde{g}_N(t,\cdot)\|_{L^2(\omega)} + \|g_N(t,\cdot) - \widetilde{g}_N(t,\cdot)\|_{L^2(\omega)}.$$

Then, since the support of $\widetilde{g}_N$ does not intersect $[0,T]\times\omega$, the first term of the right-hand side is zero, and according to the inequality (36), we have uniformly in $0 \leqslant t \leqslant T$

$$\|g_N(t,\cdot)\|^2_{L^2(\omega)} = O\big(1/N^2\big)\|\chi_N\|^2_{L^2(\mathbb{T})}.$$

Integrating this estimate for $0 \leqslant t \leqslant T$, we get the following upper bound on $\|g_N\|_{L^2([0,T]\times\omega)}$:

$$\|g_N\|^2_{L^2([0,T]\times\omega)} = O\big(1/N^2\big)\|\chi_N\|^2_{L^2(\mathbb{T})}. \tag{37}$$

To disprove the observability inequality, we also need a lower bound of $\|g_N(T,\cdot)\|_{L^2(\mathbb{T})}$. According to Parseval's identity, we have

$$\|g_N(T,\cdot)\|^2_{L^2(\mathbb{T})} = 2\pi\sum_{n\in\mathbb{Z}}|a_n^N|^2\,\Big|\mathrm{e}^{TR^{\mathrm{h}}_\mu(\mathrm{i}/n)^*}P^{\mathrm{h}}_\mu(\mathrm{i}/n)^*\varphi_0\Big|^2.$$

Since φ_0 is in the range of $P^{\mathrm{h}}_\mu(0)$, for n large enough, we have $|\mathrm{e}^{TR^{\mathrm{h}}_\mu(\mathrm{i}/n)^*}P^{\mathrm{h}}_\mu(\mathrm{i}/n)^*\varphi_0| \geqslant c > 0$. Then, since $a_n^N = 0$ for $|n| \leqslant N$, as soon as N is large enough,

$$\|g_N(T,\cdot)\|^2_{L^2(\mathbb{T})} \geqslant 2\pi c^2\sum_{n\in\mathbb{Z}}|a_n^N|^2 = 2\pi c^2\|\chi_N\|^2_{L^2(\mathbb{T})}. \tag{38}$$

Comparing the lower bound (38) and the upper bound (37), we see that the observability inequality (32) (with $M =$ identity matrix of size d) cannot hold. □

Remark 17. — The previous candidate $g_N(0,.)$ belongs to $L^2(\mathbb{T})^{d_1}\times L^2_{\mathrm{m}}(\mathbb{T})^{d_2}$ because it is supported on high frequencies $|n| > N$. Moreover, with the notation $g_N = (g_{N1}, g_{N2})$, we have

$$\frac{\|g_{N1}\|_{L^2([0,T]\times\omega)}}{\|g_N(T)\|_{L^2(\mathbb{T})}} \leqslant \frac{\|g_N\|_{L^2([0,T]\times\omega)}}{\|g_N(T)\|_{L^2(\mathbb{T})}} \underset{N\to\infty}{\longrightarrow} 0.$$

By duality, this proves the existence of $f_0 = (f_{01}, f_{02}) \in L^2(\mathbb{T})^{d_1}\times L^2_{\mathrm{m}}(\mathbb{T})^{d_2}$, such that, for any $u_1 \in L^2((0,T)\times\omega)^{d_1}$ the solution of (4) satisfies $f(T) \neq 0$.

4. Large time null-controllability

The goal of this section is to prove the point (i) of Theorem 2. An adapted decomposition of $L^2(\mathbb{T})^d$ is introduced in Section 4.1. The control strategy is presented in Section 4.2. Projecting the dynamics onto the parabolic/hyperbolic subspaces, the system is decomposed into two weakly coupled systems, the first one behaving like a transport equation, the second one like a heat equation. The transport equation is handled in Section 4.3 by using the methods developed in [1]. The parabolic equation is treated in Section 4.4 by adapting the Lebeau-Robbiano method [21] to systems with arbitrary size. The low frequency part is treated by a compactness argument and a unique continuation property in Section 4.5.

In the whole Section 4, the parameter $r > 0$ is assumed to be small enough so that Propositions 5, 6, 7, 9 and 10 hold.

4.1. An adapted decomposition of $L^2(\mathbb{T})^d$

Proposition 18. — *Let $n_0 \in \mathbb{N}^*$ be such that $1/n_0 < r$. We have the following decomposition*

$$L^2(\mathbb{T})^d = F^0 \oplus F^{\mathrm{p}} \oplus F^{\mathrm{h}}, \tag{39}$$

where

$$F^0 := \bigoplus_{|n|\leqslant n_0} \mathbb{C}^d e_n, \tag{40}$$

$$F^{\mathrm{p}} := \bigoplus_{|n|>n_0} \mathrm{Im}(P^{\mathrm{p}}(\mathrm{i}/n))e_n, \tag{41}$$

$$F^{\mathrm{h}} := \bigoplus_{|n|>n_0} \mathrm{Im}\big(P^{\mathrm{h}}(\mathrm{i}/n)\big)e_n. \tag{42}$$

Moreover the projections Π^0, Π^{p}, Π^{h} and Π defined by

$$\begin{aligned}
L^2(\mathbb{T})^d &= F^0 \oplus F^{\mathrm{p}} \oplus F^{\mathrm{h}}\\
\Pi^0 &= I_{F^0} + \ 0 \ + \ 0\\
\Pi^{\mathrm{p}} &= \ 0 \ + I_{F^{\mathrm{p}}} + \ 0\\
\Pi^{\mathrm{h}} &= \ 0 \ + \ 0 \ + I_{F^{\mathrm{h}}}\\
\Pi &= \ 0 \ + I_{F^{\mathrm{p}}} + I_{F^{\mathrm{h}}} = \Pi^{\mathrm{p}} + \Pi^{\mathrm{h}}
\end{aligned}$$

are bounded operators on $L^2(\mathbb{T})^d$.

Proof. — The function $z \in D(0,r) \mapsto P^{\mathrm{p}}(z)$ is continuous thus there exists $C > 0$ such that, for every $z \in \overline{D}(0,1/n_0)$, $|P^{\mathrm{p}}(z)| \leqslant C$. Let $f \in L^2(\mathbb{T})^d$. We deduce from

$$\sum_{|n|>n_0} \big|P^{\mathrm{p}}(\mathrm{i}/n)\widehat{f}(n)\big|^2 \leqslant C^2 \sum_{|n|>n_0} |\widehat{f}(n)|^2 \leqslant C^2\|f\|^2_{L^2(\mathbb{T})^d} \tag{43}$$

and Bessel-Parseval identity that the series $\sum P^{\mathrm{p}}(\mathrm{i}/n)\widehat{f}(n)e_n$ converges in $L^2(\mathbb{T})^d$. Using $I_d = P^{\mathrm{p}}(z) + P^{\mathrm{h}}(z)$, we get the decomposition

$$f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e_n = \sum_{|n|\leqslant n_0} \widehat{f}(n)e_n + \sum_{|n|>n_0} P^{\mathrm{p}}(\mathrm{i}/n)\widehat{f}(n)e_n + \sum_{|n|>n_0} P^{\mathrm{h}}(\mathrm{i}/n)\widehat{f}(n)e_n$$

with convergent series in $L^2(\mathbb{T})^d$. This proves $L^2(\mathbb{T})^d = F^0 + F^{\mathrm{p}} + F^{\mathrm{h}}$. The sum is direct because $(e_n)_{n\in\mathbb{Z}}$ is orthogonal and $\mathrm{Im}(P^{\mathrm{p}}(z)) \cap \mathrm{Im}(P^{\mathrm{h}}(z)) = \{0\}$ when $|z| < r$. The linear mappings Π^0 and Π are orthogonal projections, thus bounded operators on $L^2(\mathbb{T})^d$. We deduce from Bessel-Parseval identity and (43) that Π^{p} is a bounded operator on $L^2(\mathbb{T})^d$ and so is $\Pi^{\mathrm{h}} = \Pi - \Pi^{\mathrm{p}}$. □

The operator $\mathscr{L}$ defined in (21) maps $D(\mathscr{L}) \cap F^0 = F^0$ into F^0 thus we can define an operator $\mathscr{L}^0$ on F^0 by $D(\mathscr{L}^0) = D(\mathscr{L}) \cap F^0$ and $\mathscr{L}^0 = \mathscr{L}|_{F^0}$. Moreover, $-\mathscr{L}^0$ generates a C^0-semi-group of bounded operators on F^0 and $\mathrm{e}^{-t\mathscr{L}^0} = \mathrm{e}^{-t\mathscr{L}}|_{F^0}$. For the same reasons, we can define an operator $\mathscr{L}^{\mathrm{p}}$ on F^{p} by $D(\mathscr{L}^{\mathrm{p}}) = D(\mathscr{L}) \cap F^{\mathrm{p}}$, $\mathscr{L}^{\mathrm{p}} = \mathscr{L}|_{F^{\mathrm{p}}}$, $-\mathscr{L}^{\mathrm{p}}$ generates a C^0-semi-group of bounded operators on

F^{p}: $\mathrm{e}^{-t\mathscr{L}^{\mathrm{p}}} = \mathrm{e}^{-t\mathscr{L}}|_{F^{\mathrm{p}}}$. Finally, we can define an operator $\mathscr{L}^{\mathrm{h}}$ on F^{h} by $D(\mathscr{L}^{\mathrm{h}}) = D(\mathscr{L}) \cap F^{\mathrm{h}}$, $\mathscr{L}^{\mathrm{h}} = \mathscr{L}|_{F^{\mathrm{h}}}$, $-\mathscr{L}^{\mathrm{h}}$ generates a C^0-semi-group of bounded operators on F^{h}: $\mathrm{e}^{-t\mathscr{L}^{\mathrm{h}}} = \mathrm{e}^{-t\mathscr{L}}|_{F^{\mathrm{h}}}$.

Proposition 19. — *The operator $-\mathscr{L}^0$ generates a C^0 group $(\mathrm{e}^{-t\mathscr{L}^0})_{t\in\mathbb{R}}$ of bounded operators on F^0. The operator $-\mathscr{L}^{\mathrm{h}}$ generates a C^0 group $(\mathrm{e}^{-t\mathscr{L}^{\mathrm{h}}})_{t\in\mathbb{R}}$ of bounded operators on F^{h}*

Proof. — We just need to check that $\mathrm{e}^{-t\mathscr{L}}$ defines a bounded operator of F^0 and F^{h} when $t < 0$. It is clear for F^0 because it has finite dimension. For F^{h}, one may proceed as in the proof of Proposition 11, noticing that the estimate of Proposition 10 is valid for any $t \in \mathbb{R}$. □

For the duality method, we will need the dual decomposition of (39), i.e.,

$$L^2(\mathbb{T})^d = F^0 \oplus \widetilde{F^{\mathrm{p}}} \oplus \widetilde{F^{\mathrm{h}}}, \quad \text{where } \widetilde{F^{\mathrm{p}}} := \operatorname{Im}\big((\Pi^{\mathrm{p}})^*\big),\ \widetilde{F^{\mathrm{h}}} := \operatorname{Im}\big((\Pi^{\mathrm{h}})^*\big). \tag{44}$$

By using the definitions of F^{p} and F^{h} in (41) and (42) and the fact that $(e_n)_{n\in\mathbb{Z}}$ is a Hilbert basis of $L^2(\mathbb{T})$, we get

$$\widetilde{F^{\mathrm{p}}} = \bigoplus_{|n|>n_0} \operatorname{Im}(P^{\mathrm{p}}(\mathrm{i}/n)^*)e_n, \tag{45}$$

$$\widetilde{F^{\mathrm{h}}} = \bigoplus_{|n|>n_0} \operatorname{Im}\big(P^{\mathrm{h}}(\mathrm{i}/n)^*\big)e_n. \tag{46}$$

Moreover,

$$(\mathrm{e}^{-t\mathscr{L}})^* f = \mathrm{e}^{-t\mathscr{L}^*} f = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-tn^2 E(\mathrm{i}/n)^*} \widehat{f}(n) e_n \tag{47}$$

and the spaces F^0, $\widetilde{F^{\mathrm{p}}}$ and $\widetilde{F^{\mathrm{h}}}$ are stable by $\mathrm{e}^{t\mathscr{L}^*}$.

4.2. Control strategy.

— Let T^* be as in (3) and T, T' be such that

$$T^* < T' < T. \tag{48}$$

In this section, we consider controls u of the form

$$u := (u_{\mathrm{h}}, u_{\mathrm{p}})^{\mathrm{tr}} \in \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}, \tag{49}$$

where

$$\begin{aligned} &\operatorname{supp}(u_{\mathrm{h}}) \subset [0,T'] \times \overline{\omega}, && \operatorname{supp}(u_{\mathrm{p}}) \subset [T',T] \times \overline{\omega}, \\ &u_{\mathrm{h}} \in L^2((0,T') \times \mathbb{T})^{d_1}, && u_{\mathrm{p}} \in L^2((T',T) \times \mathbb{T})^{d_2}. \end{aligned} \tag{50}$$

The control u_{h} is intended to control the hyperbolic component of the system and the control u_{p} the parabolic component.

The control strategy for system (Sys) consists in

– first proving the null-controllability in time T in a subspace of $L^2(\mathbb{T})^d$ with finite codimension,

– then using a unique continuation argument, to get the full null-controllability.

The first step of this strategy is given by the following statement.

Proposition 20. — *There exist a closed subspace $\mathscr{G}$ of $L^2(\mathbb{T})^d$ with finite codimension and a continuous operator*

$$\begin{aligned}\mathscr{U} : \mathscr{G} &\longrightarrow L^2((0,T')\times\omega)^{d_1}\times C_c^\infty((T',T)\times\omega)^{d_2}\\ f_0 &\longmapsto (u_{\mathrm{h}},u_{\mathrm{p}}),\end{aligned}$$

that associates with each $f_0\in\mathscr{G}$ a pair of controls $\mathscr{U} f_0=(u_{\mathrm{h}},u_{\mathrm{p}})$ such that

$$\forall f_0\in\mathscr{G},\ \Pi S(T;f_0,\mathscr{U} f_0)=0. \tag{51}$$

By "continuous operator", we mean that, for every $s\in\mathbb{N}$, the map $\mathscr{U}:\mathscr{G}\mapsto L^2((0,T')\times\omega)^{d_1}\times H_0^s((T',T)\times\omega)^{d_2}$ is continuous: there exists $C_s>0$ such that

$$\forall f_0\in\mathscr{G},\quad \|u_{\mathrm{h}}\|_{L^2((0,T')\times\omega)^{d_1}}+\|u_{\mathrm{p}}\|_{H_0^s((T',T)\times\omega)^{d_2}}\leqslant C_s\|f_0\|_{L^2(\mathbb{T})^d}.$$

The proof strategy of Proposition 20 consists in splitting the problem in two parts:

– for any initial data f_0 and parabolic control u_{p}, steer the hyperbolic high frequencies to zero at time T (Proposition 21),

– for any initial data f_0 and hyperbolic control u_{h}, steer the parabolic high frequencies to zero at time T (Proposition 22).

Proposition 21. — *If n_0 (in* (40)–(41)*) is large enough, there exists a continuous operator*

$$\begin{aligned}\mathscr{U}^{\mathrm{h}}: L^2(\mathbb{T})^d\times L^2((T',T)\times\omega)^{d_2} &\longrightarrow L^2((0,T')\times\omega)^{d_1}\\ (f_0,u_{\mathrm{p}}) &\longmapsto u_{\mathrm{h}},\end{aligned}$$

such that for every $(f_0,u_{\mathrm{p}})\in L^2(\mathbb{T})^d\times L^2((T',T)\times\omega)^{d_2}$,

$$\Pi^{\mathrm{h}}S(T;f_0,(\mathscr{U}^{\mathrm{h}}(f_0,u_{\mathrm{p}}),u_{\mathrm{p}}))=0.$$

Proposition 22. — *If n_0 is large enough, there exists a continuous operator*

$$\begin{aligned}\mathscr{U}^{\mathrm{p}}: L^2(\mathbb{T})^d\times L^2((0,T')\times\omega)^{d_1} &\longrightarrow C_c^\infty((T',T)\times\omega)^{d_2}\\ (f_0,u_{\mathrm{h}}) &\longmapsto u_{\mathrm{p}},\end{aligned}$$

such that for every $(f_0,u_{\mathrm{h}})\in L^2(\mathbb{T})^d\times L^2((0,T')\times\omega)^{d_1}$,

$$\Pi^{\mathrm{p}}S(T;f_0,(u_{\mathrm{h}},\mathscr{U}^{\mathrm{p}}(f_0,u_{\mathrm{h}})))=0.$$

Admitting that Propositions 21 and 22 hold, we can now prove Proposition 20.

Proof. — We observe that the relation $\Pi S(T;f_0,(u_{\mathrm{h}},u_{\mathrm{p}}))=0$ holds if the two following equations are simultaneously satisfied

$$\begin{aligned}u_{\mathrm{h}}&=\mathscr{U}^{\mathrm{h}}(f_0,u_{\mathrm{p}})=\mathscr{U}_1^{\mathrm{h}}(f_0)+\mathscr{U}_2^{\mathrm{h}}(u_{\mathrm{p}}),\\ u_{\mathrm{p}}&=\mathscr{U}^{\mathrm{p}}(f_0,u_{\mathrm{h}})=\mathscr{U}_1^{\mathrm{p}}(f_0)+\mathscr{U}_2^{\mathrm{p}}(u_{\mathrm{h}}).\end{aligned} \tag{52}$$

If we set

$$C:=\mathscr{U}_1^{\mathrm{p}}+\mathscr{U}_2^{\mathrm{p}}\mathscr{U}_1^{\mathrm{h}}:L^2(\mathbb{T})^d\longrightarrow C_c^\infty((T',T)\times\mathbb{T})^{d_2},$$

then solving system (52) is equivalent to

$$\text{find } u_{\mathrm{p}}\in C_c^\infty((T',T)\times\mathbb{T})^{d_2},\ \text{such that } Cf_0=(I-\mathscr{U}_2^{\mathrm{p}}\mathscr{U}_2^{\mathrm{h}})u_{\mathrm{p}}. \tag{53}$$

The operator $\mathscr{U}_2^{\mathrm{p}}\mathscr{U}_2^{\mathrm{h}}$ is compact on $L^2((T',T)\times\mathbb{T})^{d_2}$ because it takes values in $C_c^\infty((T',T)\times\mathbb{T})^{d_2}$. Thus, by Fredhlom's alternative (see [8, Th. 6.6]), there exist $N\in\mathbb{N}$ and $\ell_1,\dots,\ell_N$ continuous linear forms on $L^2((T',T)\times\mathbb{T})^{d_2}$ such that the equation (53) has a solution $u_{\mathrm{p}}\in L^2((T',T)\times\mathbb{T})^{d_2}$ if and only if

$$\forall\, j\in\{1,\dots,N\},\ \ell_j(C(f_0))=0. \tag{54}$$

Under these conditions (54), the equation (53) has a solution $u_{\mathrm{p}}=L(f_0)$ given by a continuous map $L\colon\mathscr{G}\to L^2((T',T)\times\mathbb{T})^{d_2}$ defined on the closed vector subspace of $L^2(\mathbb{T})^d$ defined by

$$\mathscr{G}:=\{f_0\in L^2(\mathbb{T})^d\ ;\ \ell_j(Cf_0)=0,\ 1\leqslant j\leqslant N\}. \tag{55}$$

Then $L(f_0)=u_{\mathrm{p}}=\mathscr{U}_2^{\mathrm{p}}\mathscr{U}_2^{\mathrm{h}}u_{\mathrm{p}}+Cf_0$ belongs to $C_c^\infty((T',T)\times\omega)$. We get the conclusion with

$$\forall\, f_0\in\mathscr{G},\ \mathscr{U}(f_0):=(\mathscr{U}^{\mathrm{h}}(f_0,L(f_0)),L(f_0)). \qquad\square$$

Proposition 21 is proved in Section 4.3. Proposition 22 is proved in Section 4.4. The unique continuation argument to control the low frequencies is presented in Section 4.5.

4.3. Control of the hyperbolic high frequencies.

— The goal of this subsection is to prove Proposition 21. We remind that $T>T'>T^*$ and that the control $u=(u_{\mathrm{h}},u_{\mathrm{p}})$ satisfies (50).

4.3.1. *Reduction to an exact controllability problem.*

— The goal of this paragraph is to transform the null-controllability problem of Proposition 21 into an exact controllability problem associated with a hyperbolic system. Precisely, we will get Proposition 21 as a corollary of the following result.

Proposition 23. — *If n_0 (in (40)–(41)) is large enough, then, for every $T'>T^*$, there exists a continuous operator*

$$\begin{aligned}\underline{\mathscr{U}}^{\mathrm{h}}_{T'}\colon F^{\mathrm{h}}&\longrightarrow L^2((0,T')\times\omega)^{d_1}\\ f_{T'}&\longmapsto u_{\mathrm{h}},\end{aligned}$$

such that for every $f_{T'}\in F^{\mathrm{h}}$,

$$\Pi^{\mathrm{h}}S\Big(T';0,(\underline{\mathscr{U}}^{\mathrm{h}}_{T'}(f_{T'}),0)\Big)=f_{T'}.$$

Proposition 23 will be proved in Section 4.3.2. Now, we prove Proposition 21 thanks to Proposition 23.

Proof of Proposition 21. — Let $(f_0,u_{\mathrm{p}})\in L^2(\mathbb{T})^d\times L^2((T',T)\times\omega)^{d_2}$. We have to find $u_{\mathrm{h}}\in L^2((0,T')\times\omega)^{d_1}$ such that

$$\Pi^{\mathrm{h}}S(T;f_0,(u_{\mathrm{h}},u_{\mathrm{p}}))=0,$$

or, equivalently,

$$\Pi^{\mathrm{h}}S(T;0,(u_{\mathrm{h}},0))=-\Pi^{\mathrm{h}}S(T;f_0,(0,u_{\mathrm{p}})). \tag{56}$$

According to the well-posedness of the system (Sys) and the continuity of the projection Π^{h} (Definition 12 and Proposition 18), the linear map

$$(f_0, u_{\mathrm{p}}) \longmapsto -\Pi^{\mathrm{h}} S(T; f_0, (0, u_{\mathrm{p}})), \tag{57}$$

is continuous from $L^2(\mathbb{T})^d \times L^2((T', T)\times\omega)^{d_2}$ into F^{h}, equipped with the $L^2(\mathbb{T})^d$-norm. Since u_{h} is supported in $(0, T')\times\omega$ by (50), we have

$$\Pi^{\mathrm{h}} S(T; 0, (u_{\mathrm{h}}, 0)) = \mathrm{e}^{-(T-T')\mathscr{L}^{\mathrm{h}}}\Pi^{\mathrm{h}} S(T'; 0, (u_{\mathrm{h}}, 0)). \tag{58}$$

As pointed out in Proposition 19, $\mathrm{e}^{-t\mathscr{L}^{\mathrm{h}}}$ is well-defined for all $t\in\mathbb{R}$. Therefore, by using (57) and (58), (56) is equivalent to

$$\Pi^{\mathrm{h}} S(T'; 0, (u_{\mathrm{h}}, 0)) = -\mathrm{e}^{(T-T')\mathscr{L}^{\mathrm{h}}}\Pi^{\mathrm{h}} S(T; f_0, (0, u_{\mathrm{p}})) \in F^{\mathrm{h}}. \tag{59}$$

We get the conclusion with

$$\mathscr{U}^{\mathrm{h}}(f_0, u_p) = \underline{\mathscr{U}}^{\mathrm{h}}_{T'}\big(-\mathrm{e}^{(T-T')\mathscr{L}^h}\Pi^{\mathrm{h}} S(T; f_0, (0, u_{\mathrm{p}}))\big). \qquad \square$$

4.3.2. *Exact controllability of the hyperbolic part*. — The goal of this section is to prove Proposition 23. By the Hilbert Uniqueness Method, Proposition 23 is equivalent to the following observability inequality (it is an adaptation of [12, Th. 2.42]).

Proposition 24. — *If n_0 is large enough, there exists a constant $C>0$ such that for every $g_0 \in \widetilde{F^{\mathrm{h}}}$, the solution g of* (31) *satisfies*

$$\|g_0\|^2_{L^2(\mathbb{T})^d} \leqslant C\int_0^{T'}\int_\omega |g_1(t,x)|^2\,\mathrm{d}t\,\mathrm{d}x, \tag{60}$$

where g_1 denotes the first d_1 components of g.

Proof. — Let $g_0 \in \widetilde{F^{\mathrm{h}}}$. By using the definition (46) of $\widetilde{F^{\mathrm{h}}}$, and Proposition 7, g_0 decomposes as follows[6]

$$g_0 = \sum_{\mu\in\mathrm{Sp}(A')}\sum_{|n|>n_0} P^{\mathrm{h}}_\mu(\mathrm{i}/n)^*\widehat{g}_0(n)e_n. \tag{61}$$

Then, the solution g of (31) is

$$g(t) = \sum_{\mu\in\mathrm{Sp}(A')} G_\mu(t), \quad \text{where} \quad G_\mu(t) = \sum_{|n|>n_0}\mathrm{e}^{-tn^2E(\mathrm{i}/n)^*}P^{\mathrm{h}}_\mu(\mathrm{i}/n)^*\widehat{g}_0(n)e_n. \tag{62}$$

Let $\mu\in\mathrm{Sp}(A')$.

[6]Let us recall that A' is the matrix multiplying the derivative of the hyperbolic components in the system (Sys) (see (1)).

Step 1. — We prove the existence of $C_1 = C_1(T') > 0$, independent of g_0, such that

$$\|G_\mu(0,\cdot)\|_{L^2(\mathbb{T})^d} \leqslant C_1\big(\|G_\mu\|_{L^2(q_{T'})^d} + \|g_0\|_{H^{-1}(\mathbb{T})^d}\big), \tag{63}$$

where $q_{T'} = (0,T') \times \omega$ and

$$\|g_0\|_{H^{-1}(\mathbb{T})^d} = \bigg(\sum_{|n|>n_0} \frac{|\widehat{g}_0(n)|^2}{n^2}\bigg)^{1/2}. \tag{64}$$

By using (i) and (iii) of Proposition 7, we have

$$\mathrm{e}^{-tn^2E(\mathrm{i}/n)^*}P_\mu^{\mathrm{h}}(\mathrm{i}/n)^* = \mathrm{e}^{-tn^2\left(\mu\mathrm{i}/n+(\mathrm{i}/n)^2R_\mu^{\mathrm{h}}(\mathrm{i}/n)\right)^*}P_\mu^{\mathrm{h}}(\mathrm{i}/n)^* = \mathrm{e}^{t\mu\mathrm{i}n+tR_\mu^{\mathrm{h}}(\mathrm{i}/n)^*}P_\mu^{\mathrm{h}}(\mathrm{i}/n)^*,$$

which leads to

$$\partial_t G_\mu - \mu\partial_x G_\mu - R_\mu^{\mathrm{h}}(0)^* G_\mu = S_\mu \quad \text{in } (0,T')\times\mathbb{T}, \tag{65}$$

where

$$S_\mu(t) = \sum_{|n|>n_0}\big(R_\mu^{\mathrm{h}}(\mathrm{i}/n)^* - R_\mu^{\mathrm{h}}(0)^*\big)\mathrm{e}^{t\mu\mathrm{i}n+tR_\mu^{\mathrm{h}}(\mathrm{i}/n)^*}P_\mu^{\mathrm{h}}(\mathrm{i}/n)^*\widehat{g}_0(n)e_n. \tag{66}$$

By regularity of $z \mapsto R_\mu^{\mathrm{h}}(z)$, Bessel-Parseval identity and (64) there exists $C = C(T') > 0$, independent of g_0, such that

$$\|S_\mu\|_{L^\infty((0,T'),L^2(\mathbb{T})^d)} \leqslant C\|g_0\|_{H^{-1}(\mathbb{T})^d}. \tag{67}$$

By (65), the function $\widetilde{G}_\mu$ defined by

$$\widetilde{G}_\mu(t,x) = \mathrm{e}^{tR_\mu^{\mathrm{h}}(0)^*}G_\mu(t,x) \tag{68}$$

solves

$$\begin{cases}\partial_t\widetilde{G}_\mu - \mu\partial_x\widetilde{G}_\mu = \mathrm{e}^{tR_\mu^{\mathrm{h}}(0)^*}S_\mu & \text{in } (0,T')\times\mathbb{T},\\ \widetilde{G}_\mu(0,\cdot) = G_\mu(0,\cdot) & \text{in } \mathbb{T}.\end{cases} \tag{69}$$

We introduce the solution $G_\mu^\flat$ of

$$\begin{cases}\partial_t G_\mu^\flat - \mu\partial_x G_\mu^\flat = 0 & \text{in } (0,T')\times\mathbb{T},\\ G_\mu^\flat(0,\cdot) = G_\mu(0,\cdot) & \text{in } \mathbb{T}.\end{cases} \tag{70}$$

Using the Duhamel formula for the system (69) and the estimate (67), we obtain

$$\|\widetilde{G}_\mu - G_\mu^\flat\|_{L^\infty((0,T'),L^2(\mathbb{T})^d)} \leqslant C\|\mathrm{e}^{tR_\mu^{\mathrm{h}}(0)^*}S_\mu\|_{L^1((0,T'),L^2(\mathbb{T})^d)} \leqslant C\|g_0\|_{H^{-1}(\mathbb{T})^d}, \tag{71}$$

where $C = C(T') > 0$ is independent of g_0. The time $T_\mu := \ell(\omega)/|\mu|$ is the minimal time for the observability of the system (70) on ω (see for instance [1, Th. 2.2]). Indeed, for any $T'' > T_\mu$,

$$\mathbb{T} \subset \{x - \mu t;\ (t,x) \in [0,T'']\times\omega\}.$$

Since $T' > T_\mu$, there exists $C = C(T',\omega) > 0$, independent of g_0, such that

$$\|G_\mu(0,\cdot)\|_{L^2(\mathbb{T})^d} \leqslant C\|G_\mu^\flat\|_{L^2(q_{T'})^d}.$$

By the triangular inequality, (68) and (71), we deduce that

$$\begin{aligned}\|G_\mu(0,\cdot)\|_{L^2(\mathbb{T})^d} &\leqslant C\big(\|\widetilde{G}_\mu\|_{L^2(q_{T'})^d} + \|\widetilde{G}_\mu - G^\flat_\mu\|_{L^2(q_{T'})^d}\big)\\ &\leqslant C\big(\|G_\mu\|_{L^2(q_{T'})^d} + \|g_0\|_{H^{-1}(\mathbb{T})^d}\big),\end{aligned}$$

which ends the first step.

Step 2. — We prove the existence of $C_2=C_2(T',\omega)>0$, independent of g_0, such that

$$\|G_\mu(0,\cdot)\|_{L^2(\mathbb{T})^d} \leqslant C_2\big(\|P^{\mathrm{h}}_\mu(0)^*g\|_{L^2(q_{T'})^d} + \|g_0\|_{H^{-1}(\mathbb{T})^d}\big). \tag{72}$$

Taking into account that the projection $P^{\mathrm{h}}_\lambda(z)$ commutes with $E(z)$ we deduce from (62) that for any $\lambda\in\mathrm{Sp}(A')$,

$$G_\lambda(t) = \sum_{|n|>n_0} P^{\mathrm{h}}_\lambda(\mathrm{i}/n)^*\mathrm{e}^{-tn^2E(\mathrm{i}/n)^*}P^{\mathrm{h}}_\lambda(\mathrm{i}/n)^*\widehat{g}_0(n)e_n,$$

thus,

$$\begin{aligned}G_\mu(t) - P^{\mathrm{h}}_\mu(0)^*g(t) &\\ = \sum_{|n|>n_0}&\big(P^{\mathrm{h}}_\mu(\mathrm{i}/n)^* - P^{\mathrm{h}}_\mu(0)^*\big)\mathrm{e}^{-tn^2E(\mathrm{i}/n)^*}P^{\mathrm{h}}_\mu(\mathrm{i}/n)^*\widehat{g}_0(n)e_n\\ - \sum_{\lambda\in\mathrm{Sp}(A')\smallsetminus\{\mu\}}\sum_{|n|>n_0}&P^{\mathrm{h}}_\mu(0)^*\big(P^{\mathrm{h}}_\lambda(\mathrm{i}/n)^* - P^{\mathrm{h}}_\lambda(0)^*\big)\mathrm{e}^{-tn^2E(\mathrm{i}/n)^*}P^{\mathrm{h}}_\lambda(\mathrm{i}/n)^*\widehat{g}_0(n)e_n\end{aligned} \tag{73}$$

because, for $\lambda\neq\mu$, $P^{\mathrm{h}}_\mu(0)^*P^{\mathrm{h}}_\lambda(0)^*=0$. By using the regularity of $z\mapsto P^{\mathrm{h}}_\lambda(z)$, Bessel-Parseval identity and (64), we obtain the existence of $C=C(T')>0$ independent of g_0 such that

$$\|G_\mu - P^{\mathrm{h}}_\mu(0)^*g\|_{L^\infty((0,T'),L^2(\mathbb{T})^d)} \leqslant C\|g_0\|_{H^{-1}(\mathbb{T})^d}.$$

We deduce from Step 1, the triangular inequality and the previous estimate that

$$\begin{aligned}\|G_\mu(0,\cdot)\|_{L^2(\mathbb{T})^d} &\leqslant C\big(\|G_\mu\|_{L^2(q_{T'})} + \|g_0\|_{H^{-1}(\mathbb{T})^d}\big)\\ &\leqslant C\big(\|P^{\mathrm{h}}_\mu(0)^*g\|_{L^2(q_{T'})} + \|G_\mu - P^{\mathrm{h}}_\mu(0)^*g\|_{L^2(q_{T'})} + \|g_0\|_{H^{-1}(\mathbb{T})^d}\big)\\ &\leqslant C\big(\|P^{\mathrm{h}}_\mu(0)^*g\|_{L^2(q_{T'})} + \|g_0\|_{H^{-1}(\mathbb{T})^d}\big),\end{aligned}$$

which ends Step 2.

Step 3: Conclusion. — For every $\mu\in\mathrm{Sp}(A')$, we have $P^{\mathrm{h}}_\mu(0)^* = P^{\mathrm{h}}_\mu(0)^*P^{\mathrm{h}}(0)^*$ thus

$$\|P^{\mathrm{h}}_\mu(0)^*g\|_{L^2(q_{T'})} \leqslant |P^{\mathrm{h}}_\mu(0)^*|\,\|P^{\mathrm{h}}(0)^*g\|_{L^2(q_{T'})} \leqslant C\|g_1\|_{L^2(q_{T'})}.$$

Using (62), the triangular inequality, Step 2 and the previous inequality, we obtain

$$\|g_0\|_{L^2(\mathbb{T})^d} \leqslant \sum_{\mu\in\mathrm{Sp}(A')}\|G_\mu(0,\cdot)\|_{L^2(\mathbb{T})^d} \leqslant C\big(\|g_1\|_{L^2(q_{T'})^d} + \|g_0\|_{H^{-1}(\mathbb{T})^d}\big). \tag{74}$$

From this estimate and the compact embedding $L^2(\mathbb{T})\hookrightarrow H^{-1}(\mathbb{T})$, a classical compactness-uniqueness argument gives the observability inequality (60) (see for instance [13, Lem. 2.1 and Rem. 2.2]).

Indeed, by Peetre's lemma (see [24, Lem. 3]), we deduce from (74) that

$$N_{T'} := \{g_0 \in \widetilde{F^{\mathrm{h}}};\ g_1 = 0 \text{ in } (0,T') \times \omega\},$$

has finite-dimension. Moreover, from [24, Lem. 4], to prove (60), we only need to show that $N_{T'}$ is reduced to zero.

First, by definition, we remark that $N_{T'}$ decreases as T' increases. So, the map $T' \mapsto \dim(N_{T'})$ is decreasing and takes integer values. As a consequence the discontinuities on $(T^*, +\infty)$ are isolated. If T' is not such a discontinuity point, then there exists $\delta > 0$ such that $\dim(N_{T'}) = \dim(N_T)$ for every $|T - T'| < \delta$. In case T' is such a discontinuity point, one may replace T' by a smaller value, still such that $T' > T^*$, for which this holds.

By a small perturbation of T', we may therefore assume that $N_T = N_{T'}$ for $T - T'$ small, in which case $N_{T'}$ is stable by $\mathrm{e}^{-t\mathscr{L}^{*\mathrm{h}}}$ where $\mathscr{L}^{*\mathrm{h}}$ is the restriction of $\mathscr{L}^*$ to $\widetilde{F^{\mathrm{h}}}$. Then, if $N_{T'}$ is not reduced to zero, it contains an eigenfunction of $\mathscr{L}^{*\mathrm{h}}$, i.e., a function of the form Xe_n where $X \in \mathbb{C}^d$, $|n| > n_0$ and $X = P^{\mathrm{h}}(\mathrm{i}/n)X$. By definition of $N_{T'}$, the first components of that eigenfunction vanish on ω i.e., $X_1 = 0$, or equivalently $P^{\mathrm{h}}(0)X = 0$. Thus

$$|X| = \big|\big(P^{\mathrm{h}}(\mathrm{i}/n) - P^{\mathrm{h}}(0)\big)X\big| \leqslant \frac{C}{|n|}\,|X|,$$

where $C > 0$ does not depend on n. For a large enough choice of n_0, this is impossible. □

4.4. Control of the parabolic high frequencies.

— The goal of this subsection is to prove Proposition 22. We recall that T and T' are chosen such that $T^* < T' < T$ and the control u is such that (49) and (50) hold.

The strategy is the following one: identify the equation satisfied by the last d_2 components of the parabolic equation (31) with the help of the asymptotics of Proposition 7, then construct smooth controls by adapting Lebeau-Robbiano's method to systems.

In this section, for every vector $\varphi \in \mathbb{C}^d$, we will denote φ_1 its first d_1 components and φ_2 its last d_2 components.

4.4.1. *Reduction to a null-controllability problem.*

— The goal of this paragraph is to transform the null-controllability problem of Proposition 22 into a null-controllability problem associated to a parabolic system. Precisely, we will prove that Proposition 22 is a consequence of the following result.

Proposition 25. — *If n_0 is large enough, then for every $T > 0$, there exists a continuous operator*

$$\begin{aligned} \underline{\mathscr{U}}^{\mathrm{p}}_T \colon F^{\mathrm{p}} &\longrightarrow C^\infty_c((0,T)\times\omega)^{d_2} \\ f_0 &\longmapsto u_{\mathrm{p}}, \end{aligned}$$

such that for every $f_0 \in F^{\mathrm{p}}$,

$$\Pi^{\mathrm{p}} S(T; f_0, (0, \underline{\mathscr{U}}^{\mathrm{p}}_T(f_0))) = 0.$$

Proposition 25 will be proved thanks to an adaptation of Lebeau-Robbiano's method in Section 4.4.4, after two sections of preliminary results. We now prove Proposition 22 thanks to Proposition 25.

Proof of Proposition 22. — Let $(f_0, u_{\mathrm{h}}) \in L^2(\mathbb{T})^d \times L^2((0,T') \times \omega)^{d_1}$. We have to find $u_{\mathrm{p}} \in C_c^\infty((T',T) \times \omega)^{d_2}$ such that

$$\Pi^{\mathrm{p}} S(T; f_0, (u_{\mathrm{h}}, u_{\mathrm{p}})) = 0, \tag{75}$$

or equivalently,

$$\Pi^{\mathrm{p}} S(T; 0, (0, u_{\mathrm{p}})) = -\Pi^{\mathrm{p}} S(T; f_0, (u_{\mathrm{h}}, 0)). \tag{76}$$

In view of the support of the controls (Equation (50)), the equality (76) is equivalent to

$$\Pi^{\mathrm{p}} S(T-T'; 0, (0, u_{\mathrm{p}}(\cdot + T'))) = -\mathrm{e}^{-(T-T')\mathscr{L}^{\mathrm{p}}} \Pi^{\mathrm{p}} S(T'; f_0, (u_{\mathrm{h}}, 0)), \tag{77}$$

or

$$\Pi^{\mathrm{p}} S\big(T-T'; \Pi^{\mathrm{p}} S(T'; f_0, (u_{\mathrm{h}}, 0)), (0, u_{\mathrm{p}}(\cdot + T'))\big) = 0. \tag{78}$$

By using Definition 12 and Proposition 18, we see that the mapping $(f_0, u_{\mathrm{h}}) \mapsto \Pi^{\mathrm{p}} S(T'; f_0, (u_{\mathrm{h}}, 0))$ is continuous from $L^2(\mathbb{T})^d \times L^2((0,T') \times \omega)^{d_1}$ into F^{p}. Thus we get the conclusion with

$$\forall\, t \in (T',T),\ \mathscr{U}^{\mathrm{p}}(f_0, u_{\mathrm{h}})(t) = \underline{\mathscr{U}}^{\mathrm{p}}_{(T-T')}\big(\Pi^{\mathrm{p}} S(T'; f_0, (u_{\mathrm{h}}, 0))\big)(t - T'). \qquad \square$$

4.4.2. *Equation satisfied by the parabolic components of the free system*. — We begin by proving that if g is in $\widetilde{F^{\mathrm{p}}}$ then we can compute the first d_1 components of g from the last d_2. This will allow us to write an uncoupled equation for these components.

Proposition 26. — *If z is small enough, there exists a matrix $G(z)$ such that for every $\varphi \in \mathbb{C}^d$,*

$$\varphi \in \operatorname{Im}(P^{\mathrm{p}}(z)^*) \iff \varphi_1 = G(z)\varphi_2.$$

Moreover, G is holomorphic in z and $G(0) = 0$.

Proof. — We write

$$P^{\mathrm{p}}(z)^* = \begin{pmatrix} p_{11}(z) & p_{12}(z) \\ p_{21}(z) & p_{22}(z) \end{pmatrix}.$$

Since $P^{\mathrm{p}}(z)^*$ is a projection, φ is in $\operatorname{Im}(P^{\mathrm{p}}(z)^*)$ if and only if

$$\begin{cases} p_{11}(z)\varphi_1 + p_{12}(z)\varphi_2 = \varphi_1 \\ p_{21}(z)\varphi_1 + p_{22}(z)\varphi_2 = \varphi_2. \end{cases}$$

In particular, if $\varphi \in \operatorname{Im}(P^{\mathrm{p}}(z)^*)$, then $(I_{d_1} - p_{11}(z))\varphi_1 = p_{12}(z)\varphi_2$. And since $P^{\mathrm{p}}(0)^* = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{d_2} \end{smallmatrix}\right)$ (see Proposition 6), $p_{11}(0) = 0$, and so, if z is small enough, $|p_{11}(z)| < 1$ and $I_{d_1} - p_{11}(z)$ is invertible.

In that case, $\varphi_1 = (I_{d_1} - p_{11}(z))^{-1} p_{12}(z)\varphi_2$. This proves that the map

$$\varphi \in \operatorname{Im}(P^{\mathrm{p}}(z)^*) \longmapsto \varphi_2 \in \mathbb{C}^{d_2}$$

is one-to-one. But the rank of $P^{\mathrm{p}}(z)^*$ does not depend on z (Remark 8), and so it is always d_2. So the previous map is bijective. We denote by $G(z)$ the first d_1 component of its inverse. Note that we have $G(z) = (I_{d_1} - p_{11}(z))^{-1}p_{12}(z)$. Then, if $\varphi \in \mathrm{Im}(P^{\mathrm{p}}(z)^*)$, we have

$$\varphi = (\varphi_1, \varphi_2) = (G(z)\varphi_2, \varphi_2).$$

To prove the converse, note that the inverse of $\varphi \in \mathrm{Im}(P^{\mathrm{p}}(z)^*) \mapsto \varphi_2$ is $\varphi_2 \in \mathbb{C}^{d_2} \mapsto (G(z)\varphi_2, \varphi_2)$. $\square$

Increasing n_0 if necessary, we may assume that for $|n| > n_0$, $G(\mathrm{i}/n)$ is well-defined. Then, we define the (bounded) operator G from $L^2(\mathbb{T}, \mathbb{C}^{d_2})$ to $L^2(\mathbb{T}, \mathbb{C}^{d_1})$ by

$$G\Big(\sum_{n\in\mathbb{Z}} \varphi_{n,2} e_n\Big) = \sum_{|n|>n_0} G(\mathrm{i}/n)\varphi_{n,2} e_n. \tag{79}$$

According to the definition of $\widetilde{F^{\mathrm{p}}}$, we have the following corollary that enables us to compute the first d_1 components from the last d_2.

Corollary 27. — *For every $g \in (F^0)^\perp$ (the space of functions with no components along frequencies less than n_0), we have the equivalence $g \in \widetilde{F^{\mathrm{p}}} \Leftrightarrow g_1 = Gg_2$.*

Corollary 27 makes it easy to write an equation on the last d_2 components of the adjoint system (31) if the initial condition is in $\widetilde{F^{\mathrm{p}}}$.

Proposition 28. — *We define the operator $\mathfrak{D}$ by*

$$D(\mathfrak{D}) = H^2(\mathbb{T})^{d_2}, \quad \mathfrak{D} = D^{\mathrm{tr}}\partial_x^2 + A_{22}^{\mathrm{tr}}\partial_x - K_{22}^{\mathrm{tr}} + A_{12}^{\mathrm{tr}}\partial_x G - K_{12}^{\mathrm{tr}} G. \tag{80}$$

Let $g_0 \in \widetilde{F^{\mathrm{p}}}$ and $g(t) = \mathrm{e}^{-t\mathscr{L}^*}g_0$. Then, for all $t \geqslant 0$, $g_1(t) = Gg_2(t)$ and g_2 satisfies the following equation

$$\partial_t g_2(t,x) - \mathfrak{D} g_2(t,x) = 0 \qquad \textit{in } (0,T)\times\mathbb{T}. \tag{81}$$

Proof. — The function g satisfies the system

$$(\partial_t - B^{\mathrm{tr}}\partial_x^2 - A^{\mathrm{tr}}\partial_x + K^{\mathrm{tr}})g(t,x) = 0 \qquad \text{in } (0,T)\times\mathbb{T}.$$

If we take the last d_2 components of this system, we get, in $(0,T)\times\mathbb{T}$,

$$\big(\partial_t - D^{\mathrm{tr}}\partial_x^2 - A_{22}^{\mathrm{tr}}\partial_x + K_{22}^{\mathrm{tr}}\big)g_2(t,x) - \big(A_{12}^{\mathrm{tr}}\partial_x - K_{12}^{\mathrm{tr}}\big)g_1(t,x) = 0. \tag{82}$$

But for all $t \in [0,T]$, $g(t,\cdot) \in \widetilde{F^{\mathrm{p}}}$, so, according to Corollary 27, $g_1(t) = Gg_2(t)$. Substituting this inside Equation (82) gives the stated equation (81). $\square$

4.4.3. *Smooth control of a finite number of parabolic vector components*

For $N > n_0$ we introduce

$$F^{\mathrm{p}}_N := \bigoplus_{n_0<|n|\leqslant N} \mathrm{Im}(P^{\mathrm{p}}(\mathrm{i}/n))e_n, \quad F^{\mathrm{p}}_{>N} := \bigoplus_{|n|>N} \mathrm{Im}(P^{\mathrm{p}}(\mathrm{i}/n))e_n. \tag{83}$$

and the projection Π^{p}_N defined by

$$\begin{aligned} L^2(\mathbb{T})^d &= F^0 \oplus F^{\mathrm{p}}_N \oplus F^{\mathrm{p}}_{>N} \oplus F^{\mathrm{h}} \\ \Pi^{\mathrm{p}}_N &= \ 0 \ + I_{F^{\mathrm{p}}_N} + \ \ 0 \ \ + \ 0 \end{aligned}$$

which is a bounded operator on $L^2(\mathbb{T})^d$ (composition of the bounded operator Π^{p} with an orthogonal projection). The goal of this section is to prove the following result.

Proposition 29. — *There exists $\mathscr{C} > 0$ such that, for every $T \in (0,1]$ and $N > n_0$, there exists a linear map*[7]

$$\mathscr{K}_{T,N} : F^{\mathrm{p}} \longrightarrow C^\infty_0((0,T)\times\omega)$$

such that, for every $f_0 \in F^{\mathrm{p}}$ and $s \in \mathbb{N}$

$$\begin{gathered} \Pi^{\mathrm{p}}_N S\big(T; f_0, (0, \mathscr{K}_{T,N}(f_0))\big) = 0, \\ \|\mathscr{K}_{T,N}(f_0)\|_{H^s_0((0,T)\times\mathbb{T})} \leqslant \frac{\mathscr{C}}{T^{s+1}}\, N^{2s} \mathrm{e}^{\mathscr{C}N} \|f_0\|_{L^2(\mathbb{T})^d}. \end{gathered}$$

Proof. — Let $f_0 \in F^{\mathrm{p}}$. Throughout this proof, we will denote by $E_2(n)$ the $d_2\times d_2$ matrices defined by

$$\forall\, |n| > n_0,\ E_2(n) := D^{\mathrm{tr}} - \frac{\mathrm{i}}{n} A^{\mathrm{tr}}_{22} + \frac{1}{n^2} K^{\mathrm{tr}}_{22} - \Big(\frac{\mathrm{i}}{n} A^{\mathrm{tr}}_{12} - \frac{1}{n^2} K^{\mathrm{tr}}_{12}\Big) G(\mathrm{i}/n).$$

Step 1. — We prove that $u_2 \in C^\infty_0((0,T)\times\omega)$ satisfies $\Pi^{\mathrm{p}}_N S(T; f_0, (0,u_2)) = 0$ if and only if u_2 solves the following moments problem in $\mathbb{C}^{d_2}$

$$\forall\, n_0 < |n| \leqslant N,\ \int_0^T \int_\omega \mathrm{e}^{-n^2(T-t)E_2(n)^*} u_2(t,x) \mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x\,\mathrm{d}t = F_n \tag{84}$$
$$\text{where } F_n = -\mathrm{e}^{-n^2 T E_2(n)^*}\Big(G(\mathrm{i}/n)^* \widehat{f}_{01}(n) + \widehat{f}_{02}(n)\Big)$$

and $E_2(n)^* = \overline{E_2(n)}^{\mathrm{tr}}$.

We first recall that, if P is a projection operator on $\mathbb{R}^d$ and $x \in \mathrm{Im}(P)$, then

$$(x = 0) \iff (\forall\, z \in \mathrm{Im}(P^*), \langle x, z\rangle = 0)$$

because $|x|^2 = \langle x,x\rangle = \langle Px, x\rangle = \langle x, P^*x\rangle$.

As a consequence, the relation $\Pi^{\mathrm{p}}_N S(T; f_0, (0,u_2)) = 0$ is equivalent to

$$\forall\, g_T \in \widetilde{F^{\mathrm{p}}_N},\ \langle S(T; f_0, (0,u_2)), g_T\rangle = 0, \tag{85}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product of $L^2(\mathbb{T},\mathbb{C}^d)$ and

$$\widetilde{F^{\mathrm{p}}_N} := \bigoplus_{n_0<|n|\leqslant N} \mathrm{Im}\big(P^{\mathrm{p}}(\mathrm{i}/n)^*\big) e_n.$$

[7] The space $C^\infty_0((0,T)\times\omega)$ consists of functions supported on $[0,T]\times K$, where K is a compact subset of ω, and all of whose derivatives vanish on ω at time $t=0$ and $t=T$.

For $g_T \in \widetilde{F_N^{\mathrm{p}}}$, we denote by $g(t) = \mathrm{e}^{-\mathscr{L}^*(T-t)} g_T$ the solution of the adjoint system (31). Then, by Proposition 28, $g = (g_1, g_2)$, where $g_1 = G(g_2)$ and

$$\langle S(T; f_0, (0, u_2)), g_T \rangle = \langle f_0, g(0) \rangle + \int_0^T \int_\omega \langle u_2(t,x), g_2(t,x) \rangle \, \mathrm{d}x \, \mathrm{d}t,$$

where the first two scalar products are in $L^2(\mathbb{T})^d$ and the last one is in $\mathbb{C}^{d_2}$. By Corollary 27, the assertion (85) is equivalent to

$$\forall\, g_2^T \in \operatorname{Span}\{X e_n,\ X \in \mathbb{C}^{d_2}, n_0 < |n| \leqslant N\},$$
$$\int_0^T \int_\omega \langle u_2(t,x), g_2(t,x) \rangle \, \mathrm{d}x \, \mathrm{d}t = -\big\langle f_0, \big(G(g_2^0), g_2^0\big)\big\rangle, \tag{86}$$

where $g_2(t) = \mathrm{e}^{-\mathfrak{D}(T-t)} g_2^T$ and $g_2^0 = g_2(0)$. By considering $g_2^T = X e_n$ with $X \in \mathbb{C}^{d_2}$ and $n_0 < |n| \leqslant N$, we obtain

$$g_2(t) = \mathrm{e}^{-n^2(T-t)E_2(n)} X e_n \text{ and } G(g_2^0) = G(\mathrm{i}/n) \mathrm{e}^{-n^2 T E_2(n)} X e_n.$$

The previous property is equivalent to

$$\forall\, n_0 < |n| \leqslant N,\ \forall\, X \in \mathbb{C}^{d_2},\ \int_0^T \int_\omega \langle u_2(t,x), \mathrm{e}^{-n^2(T-t)E_2(n)} X \rangle \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x \, \mathrm{d}t$$
$$= -\langle f_{01}, G(\mathrm{i}/n) \mathrm{e}^{-n^2 T E_2(n)} X e_n \rangle - \langle f_{02}, \mathrm{e}^{-n^2 T E_2(n)} X e_n \rangle$$

or, equivalently,

$$\forall\, n_0 < |n| \leqslant N,\ \forall\, X \in \mathbb{C}^{d_2},\ \Big\langle \int_0^T \int_\omega \mathrm{e}^{-n^2(T-t)E_2(n)^*} u_2(t,x) \mathrm{e}^{-\mathrm{i}nx} \, \mathrm{d}x \, \mathrm{d}t, X \Big\rangle$$
$$= -\big\langle \mathrm{e}^{-n^2 T E_2(n)^*} G(\mathrm{i}/n)^* \widehat{f}_{01}(n) + \mathrm{e}^{-n^2 T E_2(n)^*} \widehat{f}_{02}(n), X \big\rangle,$$

which proves (84).

Step 2: Solving the moment problem. — We look for a solution $u_2 \in C_0^\infty((0,T) \times \omega)$ of the moment problem (84) of the form

$$u_2(t,x) = \rho(t,x) v_2(t,x), \tag{87}$$

where $v_2 \in C^\infty((0,T) \times \mathbb{T})^{d_2}$ and $\rho \in C_0^\infty((0,T) \times \omega)$ is a scalar function with an appropriate support. More precisely, let

– $\widehat{\omega}$ be an open subset such that $\widehat{\omega} \Subset \omega$ and $\rho_2 \in C_c^\infty(\omega, \mathbb{R}_+)$ such that $\rho_2 = 1$ on $\widehat{\omega}$,

– $\rho_1 \in C^\infty([0,1], \mathbb{R}_+)$ such that $\rho_1(0) = \rho_1(1) = 0$ and

$$\exists\, C_0 > 0,\ \forall\, \gamma > 0, \quad \int_0^1 \rho_1(\tau) \mathrm{e}^{-\gamma\tau} \, \mathrm{d}\tau \geqslant \frac{1}{C_0} \mathrm{e}^{-C_0 \sqrt{\gamma}}. \tag{88}$$

For instance, we may consider ρ_1 such that $\rho_1(\tau) = \rho_1(1-\tau) = \mathrm{e}^{-1/\tau}$ for $\tau \in (0, 1/4)$. Indeed, for every $\gamma > 0$, the change of variable $s = \sqrt{\gamma}\,\tau$ gives

$$\int_0^1 \rho_1(\tau) \mathrm{e}^{-\gamma\tau} \, \mathrm{d}\tau \geqslant \frac{1}{\sqrt{\gamma}} \int_0^{\sqrt{\gamma}/4} \mathrm{e}^{-\sqrt{\gamma}\,\phi(s)} \, \mathrm{d}s,$$

where $\phi(s)=s+1/s$. The function ϕ takes its minimal value at $s_*=1$ and $\phi''(1)=2>0$ thus, by Laplace's method (see [25, Ch. 9, Th. VI.1]),

$$\int_0^2 \mathrm{e}^{-\sqrt{\gamma}\phi(s)}\,\mathrm{d}s \underset{\gamma\to\infty}{\sim} \frac{\sqrt{\pi}}{\sqrt[4]{\gamma}}\,\mathrm{e}^{-2\sqrt{\gamma}}.$$

which proves (88) for a large enough constant C_0.

Then we choose $\rho(t,x)=\rho_1((T-t)/T)\rho_2(x)$. We also look for v_2 of the form

$$v_2(t,x)=\sum_{n_0<|k|\leqslant N}\mathrm{e}^{-k^2(T-t)E_2(k)}V_k\mathrm{e}^{\mathrm{i}kx}\quad\text{where } V_k\in\mathbb{C}^{d_2}. \tag{89}$$

The construction of v_2 will use the following algebraic result.

Lemma 30. — *There exists $\mathscr{C}>0$ such that, for every $N>n_0$ and $T\in(0,1]$ the matrix A in $\mathbb{C}^{(2(N-n_0)d_2)\times(2(N-n_0)d_2)}$, defined by blocks $A=(A_{n,k})_{\substack{n_0<|n|\leqslant N\\ n_0<|k|\leqslant N}}$ by*

$$A_{n,k}=\int_0^T\int_\omega \mathrm{e}^{-n^2(T-t)E_2(n)^*}\mathrm{e}^{-k^2(T-t)E_2(k)}\mathrm{e}^{\mathrm{i}(k-n)x}\rho(t,x)\,\mathrm{d}x\,\mathrm{d}t\ \in\mathbb{C}^{d_2\times d_2},$$

is invertible and

$$\forall\,F\in\mathbb{C}^{2(N-n_0)d_2},\quad |A^{-1}F|\leqslant\frac{\mathscr{C}}{T}\,\mathrm{e}^{\mathscr{C}N}|F|,$$

where $|\cdot|$ is the hermitian norm on $\mathbb{C}^{2(N-n_0)d_2}$.

Remark 31. — For instance, when $N=n_0+2$, then A is given by

$$A=\begin{pmatrix} A_{-n_0-2,-n_0-2} & A_{-n_0-2,-n_0-1} & A_{-n_0-2,n_0+1} & A_{-n_0-2,n_0+2}\\ A_{-n_0-1,-n_0-2} & A_{-n_0-1,-n_0-1} & A_{-n_0-1,n_0+1} & A_{-n_0-1,n_0+2}\\ A_{n_0+1,-n_0-2} & A_{n_0+1,-n_0-1} & A_{n_0+1,n_0+1} & A_{n_0+1,n_0+2}\\ A_{n_0+2,-n_0-2} & A_{n_0+2,-n_0-1} & A_{n_0+2,n_0+1} & A_{n_0+2,n_0+2}\end{pmatrix}.$$

For $X\in\mathbb{C}^{4d_2}$ with block decomposition

$$X=\begin{pmatrix}X_{-n_0-2}\\ X_{-n_0-1}\\ X_{n_0+1}\\ X_{n_0+2}\end{pmatrix}$$

where $X_k\in\mathbb{C}^{d_2}$ for every $n_0<|k|\leqslant n_0+2$, we have

$$AX=\begin{pmatrix}\sum_{n_0<|k|\leqslant n_0+2}A_{-n_0-2,k}X_k\\ \sum_{n_0<|k|\leqslant n_0+2}A_{-n_0-1,k}X_k\\ \sum_{n_0<|k|\leqslant n_0+2}A_{n_0+1,k}X_k\\ \sum_{n_0<|k|\leqslant n_0+2}A_{n_0+2,k}X_k\end{pmatrix}.$$

Thus $\langle X,AX\rangle=\sum_{n_0<|n|,|k|\leqslant n_0+2}X_n^*A_{n,k}X_k$.

Proof of Lemma 30. — The proof relies on the following spectral inequality, due to Lebeau and Robbiano (see [21] and also [20, Th. 5.4]):

$$\exists\,C_1>0,\ \forall\,N\in\mathbb{N},\ \forall\,(a_n)_{n\in\mathbb{Z}}\in\mathbb{C}^{\mathbb{Z}},\quad \sum_{n=-N}^{+N}|a_n|^2\leqslant C_1\mathrm{e}^{C_1N}\int_{\widehat{\omega}}\Big|\sum_{n=-N}^{+N}a_n\mathrm{e}^{\mathrm{i}nx}\Big|^2\,\mathrm{d}x. \tag{90}$$

By summing the components, the same inequality holds when a_n is a vector, $a_n \in \mathbb{C}^{d_2}$, and $|\cdot|$ denotes the hermitian norm on $\mathbb{C}^{d_2}$.

Let $N > n_0$ and $X \in \mathbb{C}^{2(N-n_0)d_2}$ written by blocks $X = (X_k)_{n_0<|k|\leqslant N}$ with $X_k \in \mathbb{C}^{d_2}$. Then, by using the definition of A, ρ, the properties of ρ_2 and the above spectral inequality in vector form, we obtain

$$\begin{aligned}
\langle AX, X\rangle &= \sum_{n_0<|n|,|k|\leqslant N} X_n^* A_{n,k} X_k \\
&= \int_0^T \int_\omega \Big| \sum_{n_0<|k|\leqslant N} \mathrm{e}^{-k^2(T-t)E_2(k)} X_k \mathrm{e}^{\mathrm{i}kx} \Big|^2 \rho(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant \int_0^T \int_{\widehat{\omega}} \Big| \sum_{n_0<|k|\leqslant N} \mathrm{e}^{-k^2(T-t)E_2(k)} X_k \mathrm{e}^{\mathrm{i}kx} \Big|^2 \rho_1((T-t)/T)\,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant \frac{\mathrm{e}^{-C_1 N}}{C_1} \int_0^T \sum_{n_0<|k|\leqslant N} \big|\mathrm{e}^{-k^2(T-t)E_2(k)} X_k\big|^2 \rho_1((T-t)/T)\,\mathrm{d}t.
\end{aligned}$$

There exists $c > 0$ such that, for every $|k| > n_0$, $|E_2(k)| \leqslant c$. Then,

$$\forall\,|k| > n_0,\ \tau > 0,\ Y \in \mathbb{C}^{d_2}, \quad |\mathrm{e}^{E_2(k)\tau} Y| \leqslant \mathrm{e}^{c\tau}|Y|.$$

By considering $\tau = k^2(T-t)$ and $Y = \exp\big(-k^2(T-t)E_2(k)\big)X_k$, we obtain

$$\forall\,|k| > n_0,\ t \in (0,T), \quad \big|\mathrm{e}^{-k^2(T-t)E_2(k)} X_k\big| \geqslant \mathrm{e}^{-ck^2(T-t)}|X_k|.$$

Therefore, by using the change of variable $\tau = (T-t)/T$ and (88), we get

$$\begin{aligned}
\langle AX, X\rangle &\geqslant \frac{T\mathrm{e}^{-C_1 N}}{C_1} \sum_{n_0<|k|\leqslant N} |X_k|^2 \int_0^T \mathrm{e}^{-2ck^2T\tau}\rho_1(\tau)\,\mathrm{d}\tau \\
&\geqslant \frac{T\mathrm{e}^{-C_1 N}}{C_1 C_0} \sum_{n_0<|k|\leqslant N} |X_k|^2 \mathrm{e}^{-C_0 k\sqrt{2cT}} \\
&\geqslant \frac{T}{C_1 C_0}\,\mathrm{e}^{-(C_1 + C_0\sqrt{2cT})N}|X|^2.
\end{aligned}$$

The above relation, valid for any $X \in \mathbb{C}^{2(N-n_0)d_2}$ proves that any eigenvalue of A is positive, thus A is invertible. Moreover, for any $F \in \mathbb{C}^{2(N-n_0)d_2} \smallsetminus \{0\}$, the vector $X = A^{-1}F$ satisfies

$$\frac{T}{C_1 C_0}\,\mathrm{e}^{-(C_1 + C_0\sqrt{2cT})N}|X|^2 \leqslant \langle AX, X\rangle = \langle F, X\rangle \leqslant |F||X|.$$

Thus

$$|X| \leqslant \frac{C_1 C_0}{T}\,\mathrm{e}^{(C_1 + C_0\sqrt{2cT})N}|F|.$$

This gives the conclusion with $\mathscr{C} = \max\big\{C_1 C_0;\ C_1 + C_0\sqrt{2c}\big\}$. □

Now, let us come back to the proof of Proposition 29. For such a control of the form given by Equations (87) and (89), the moment problem (84) writes

$$\forall\, n_0 < |n| \leqslant N, \quad \sum_{n_0<|k|\leqslant N} A_{n,k} V_k = F_n$$

or equivalently $AV = F$ with the notation of Lemma 30. Thus, it is sufficient to take $V = A^{-1}F$. By the definition of F in (84), and Bessel-Parseval identity there exists $C_2 > 0$ independent of (T, N) such that

$$|F| = \Big(\sum_{n_0<|n|\leqslant N} |F_n|^2\Big)^{1/2} \leqslant C_2 \|f_0\|_{L^2(\mathbb{T})^d}.$$

Therefore, by Lemma 30

$$|V| = \Big(\sum_{n_0<|k|\leqslant N} |V_k|^2\Big)^{1/2} \leqslant \frac{C_2\mathscr{C}}{T}\, \mathrm{e}^{\mathscr{C}N} \|f_0\|_{L^2(\mathbb{T})^d}. \tag{91}$$

Step 3: Estimates on u_2. — Let $s \in \mathbb{N}^*$. By (87) and the definition of ρ, there exists $C = C(\rho, s) > 0$ such that

$$\|u_2\|_{H^s((0,T)\times\omega)} \leqslant \frac{C}{T^s}\, \|v_2\|_{H^s((0,T)\times\mathbb{T})}. \tag{92}$$

For any $s_1, s_2 \in \mathbb{N}$ such that $s_1 + s_2 \leqslant s$ we have,

$$\partial_t^{s_1}\partial_x^{s_2} v_2(t,x) = \sum_{n_0<|k|\leqslant N} k^{2s_1} E_2(k)^{s_1} \mathrm{e}^{-k^2(T-t)E_2(k)} V_k (\mathrm{i}k)^{s_2} \mathrm{e}^{\mathrm{i}kx}.$$

By Bessel-Parseval identity, we have

$$\begin{aligned}\|\partial_t^{s_1}\partial_x^{s_2} v_2\|^2_{L^2((0,T)\times\mathbb{T})} &= \int_0^T \sum_{n_0<|k|\leqslant N} \Big| k^{2s_1+s_2} E_2(k)^{s_1} \mathrm{e}^{-k^2(T-t)E_2(k)} V_k\Big|^2 \,\mathrm{d}t \\ &\leqslant C \int_0^T \sum_{n_0<|k|\leqslant N} k^{4s} \Big|\mathrm{e}^{-k^2(T-t)E_2(k)} V_k\Big|^2 \,\mathrm{d}t.\end{aligned}$$

By working as in the proof of Proposition 9, we obtain, for n_0 large enough, positive constants $K_p, c_p > 0$ such that

$$\begin{aligned}\|\partial_t^{s_1}\partial_x^{s_2} v_2\|^2_{L^2((0,T)\times\mathbb{T})} &\leqslant C \sum_{n_0<|k|\leqslant N} k^{4s} K_p^2 \int_0^T \mathrm{e}^{-2c_p k^2(T-t)} \,\mathrm{d}t\, |V_k|^2 \\ &\leqslant \frac{CK_p^2}{2c_p} \sum_{n_0<|k|\leqslant N} k^{4s-2} |V_k|^2 \leqslant \frac{CK_p^2}{c_p} N^{4s-1} |V|^2.\end{aligned}$$

By (91),

$$\|\partial_t^{s_1}\partial_x^{s_2} v_2\|_{L^2((0,T)\times\mathbb{T})} \leqslant \sqrt{\frac{C}{c_p}}\, K_p N^{2s-1/2}\, \frac{C_2\mathscr{C}}{T}\, \mathrm{e}^{\mathscr{C}N} \|f_0\|_{L^2(\mathbb{T})^d}.$$

This provides a constant $C>0$ independent of (T,N) such that

$$\|v_2\|_{H^s((0,T)\times\mathbb{T})} \leqslant \frac{C}{T} N^{2s-1/2}\, \mathrm{e}^{\mathscr{C}N} \|f_0\|_{L^2(\mathbb{T})^d}$$

and (92) gives the expected estimate on u in H^s. □

4.4.4. *Lebeau-Robbiano's method.* — The goal of this section is to prove Proposition 25. Let $T>0$. We fix $\delta\in(0,T/2)$ and $\rho\in(0,1)$. For $\ell\in\mathbb{N}^*$, we set $N_\ell=2^\ell$, $T_\ell = A2^{-\rho\ell}$, where $A>0$ is such that $2\sum_{\ell=1}^{\infty} T_\ell = T-2\delta$. Let $f_0\in F^{\mathrm{p}}$. We define

$$\begin{cases} f_1 = \mathrm{e}^{-\delta \mathscr{L}^{\mathrm{p}}} f_0, \\ g_\ell = \Pi^{\mathrm{p}} S(T_\ell; f_\ell, u_\ell) \text{ where } u_\ell = (0, K_{T_\ell,N_\ell}(f_\ell)), \\ f_{\ell+1} = \mathrm{e}^{-T_\ell \mathscr{L}^{\mathrm{p}}} g_\ell, \end{cases}$$

where K_{T_ℓ,N_ℓ} is the control operator introduced in Proposition 29. By construction $\Pi^{\mathrm{p}}_{N_\ell} g_\ell = 0$ and therefore, by Proposition 9,

$$\begin{aligned} \|f_{\ell+1}\|^2_{L^2(\mathbb{T})^d} = \|\mathrm{e}^{-T_\ell\mathscr{L}^{\mathrm{p}}} g_\ell\|^2_{L^2(\mathbb{T})^d} &= \sum_{|n|>N_\ell} \Big|\mathrm{e}^{-n^2 E(\mathrm{i}/n) T_\ell} \widehat{g}_\ell(n)\Big|^2 \\ &\leqslant \sum_{|n|>N_\ell} K_p^2 \mathrm{e}^{-2n^2 c_p T_\ell} |\widehat{g}_\ell(n)|^2 \leqslant K_p^2 \mathrm{e}^{-2c_p N_\ell^2 T_\ell} \|g_\ell\|^2_{L^2(\mathbb{T})^d}. \end{aligned}$$

By the semi-group property proved in Proposition 11, there exist positive constants K and c such that

$$\forall\, f\in L^2(\mathbb{T})^d,\ t\geqslant 0, \quad \|\mathrm{e}^{-t\mathscr{L}} f\|_{L^2(\mathbb{T})^d} \leqslant K\mathrm{e}^{ct}\|f\|_{L^2(\mathbb{T})^d}.$$

Then, according to the triangle inequality and Cauchy-Schwarz inequality,

$$\begin{aligned} \|g_\ell\|_{L^2(\mathbb{T})^d} \leqslant \|S(T_\ell; f_\ell, u_\ell)\| &\leqslant K\mathrm{e}^{cT_\ell}\|f_\ell\|_{L^2(\mathbb{T})^d} + \int_0^{T_\ell} K\mathrm{e}^{c(T_\ell - t)} \|u_\ell(t)\|_{L^2(\mathbb{T})}\,\mathrm{d}t \\ &\leqslant K\mathrm{e}^{cT_\ell}\big(\|f_\ell\|_{L^2(\mathbb{T})^d} + \sqrt{T_\ell}\,\|u_\ell\|_{L^2((0,T_\ell)\times\omega)}\big), \end{aligned}$$

and by Proposition 29

$$\|u_\ell\|_{L^2((0,T_\ell)\times\omega)} \leqslant \frac{\mathscr{C}}{T_\ell}\, \mathrm{e}^{\mathscr{C}N_\ell} \|f_\ell\|_{L^2(\mathbb{T})^d}. \tag{93}$$

Thus

$$\|g_\ell\|_{L^2(\mathbb{T})^d} \leqslant K\mathrm{e}^{cT_\ell}\Big(1+\frac{\mathscr{C}}{\sqrt{T_\ell}}\,\mathrm{e}^{\mathscr{C}N_\ell}\Big)\|f_\ell\|_{L^2(\mathbb{T})^d}.$$

By setting

$$m_\ell = K_p \mathrm{e}^{-c_p N_\ell^2 T_\ell} K\mathrm{e}^{cT_\ell}\Big(1+\frac{\mathscr{C}}{\sqrt{T_\ell}}\,\mathrm{e}^{\mathscr{C}N_\ell}\Big),$$

we get

$$\|f_{\ell+1}\|_{L^2(\mathbb{T})^d} \leqslant m_\ell \|f_\ell\|_{L^2(\mathbb{T})^d}.$$

It is easy to see that there exist $C_1, C_2>0$ such that $m_\ell \leqslant C_1 \mathrm{e}^{-C_2 2^{(2-\rho)\ell}}$. Thus $\|f_\ell\|_{L^2(\mathbb{T})^d}\to 0$ and more precisely there exist positive constants $C_3, C_4>0$ such that

$$\|f_\ell\|_{L^2(\mathbb{T})^d} \leqslant C_3 \exp\big(-C_4 2^{(2-\rho)\ell}\big)\|f_0\|_{L^2(\mathbb{T})^d}.$$

Moreover, from (93),

$$\sum_{\ell=1}^{\infty}\|u_\ell\|^2_{L^2((0,T_\ell)\times\omega)}\leqslant \mathscr{C}\sum_{\ell=1}^{\infty}\frac{\mathrm{e}^{\mathscr{C}N_\ell}}{}T_\ell C_3\exp(-C_4 2^{(2-\rho)\ell})\|f_0\|_{L^2(\mathbb{T})^d}<\infty. \tag{94}$$

We set $a_0=\delta$, $a_2=\delta+2T_1,\dots,a_\ell=a_{\ell-1}+2T_\ell$. We have $a_\ell\to(T-\delta)$ as $\ell\to\infty$. For any $f_0\in F^{\mathrm{p}}$, we define the control

$$\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0)(t,x)=\begin{cases}K_{T_\ell,N_\ell}(f_\ell)(t-a_{\ell-1}) & \text{for } a_{\ell-1}\leqslant t\leqslant a_{\ell-1}+T_\ell,\\ 0 & \text{for } a_{\ell-1}+T_\ell\leqslant t\leqslant a_{\ell-1}+2T_\ell=a_\ell,\\ 0 & \text{for } T-\delta\leqslant t\leqslant T.\end{cases}$$

Then, $\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0)\in C_0^\infty((\delta,T-\delta)\times\omega)^{d_2}$ because all its derivatives vanish at times $t=a_\ell$. Thus $\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0)\in C_c^\infty((0,T)\times\omega)^{d_2}$.

By (94), $\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0)\in L^2((0,T)\times\omega)^d$, thus $S(T-\delta;f_0,\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0))$ is the limit, in $L^2(\mathbb{T})^d$, as $\ell\to\infty$, of the sequence $S(a_\ell;f_0,\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0))$. As a consequence, $\Pi^{\mathrm{p}}S(T-\delta;f_0,\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0))$ is the limit in $L^2(\mathbb{T})$ of the sequence $\Pi^{\mathrm{p}}S(a_\ell;f_0,\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0))=f_{\ell+1}$. Finally,

$$\Pi^{\mathrm{p}}S(T;f_0,\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0))=\Pi^{\mathrm{p}}S(T-\delta;f_0,\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0))=0.$$

By Proposition 29, for any $s\in\mathbb{N}^*$,

$$\|\underline{\mathscr{U}}_T^{\mathrm{p}}(f_0)\|_{H^s((0,T)\times\omega)}\leqslant\underbrace{\sum_{\ell=1}^{\infty}\frac{\mathscr{C}}{T_\ell^{s+1}}N_\ell^{2s}\mathrm{e}^{\mathscr{C}N_\ell}C_3\exp\bigl(-C_4 2^{(2-\rho)\ell}\bigr)}_{<\infty}\|f_0\|_{L^2(\mathbb{T})^d}.$$

This concludes the proof of Proposition 25. □

4.5. Control of the low frequencies. — The goal of this subsection is to prove Theorem 2. Let $T>T^*$ where T^* is defined in (3). Then, there exists $T'>0$ such that (48) holds. Let $\mathscr{G}$ and $\mathscr{U}$ be as in Proposition 20.

Without loss of generality, we may assume that $F_0\subset\mathscr{G}$ by the following procedure. Let W be a complement of $\mathscr{G}\cap F^0$ in F^0. Then W is a complement of $\mathscr{G}$ in[8] $\mathscr{G}+F^0$, and we extend $\mathscr{U}$ to $\mathscr{G}\oplus W$ by setting $\mathscr{U}(f_0)=0$ for every $f_0\in W$.

Implicitly, $\mathscr{G}$ is equipped with the topology of the $L^2(\mathbb{T})^d$-norm. The operator S is defined in Definition 12.

We introduce the vector subspace of $L^2(\mathbb{T})^d$ defined by

$$\mathscr{F}_T=\bigl\{f_0\in L^2(\mathbb{T})^d;\ \exists\, u\in L^2((0,T')\times\omega)^{d_1}\times C_c^\infty((T',T)\times\omega)^{d_2}/S(T;f_0,u)=0\bigr\}.$$

[8] If $f\in\mathscr{G}+F^0$, we write it as $f_{\mathscr{G}}+f_{F_0}$, and in turn we decompose f_{F_0} along the sum $F_0=\mathscr{G}\cap F_0\oplus W$: $f_{F_0}=f_{\mathscr{G}\cap F_0}+f_W\in\mathscr{G}+W$. So $f=(f_{\mathscr{G}}+f_{\mathscr{G}\cap F_0})+f_W$. This proves that $\mathscr{G}+F_0=\mathscr{G}+W$. Moreover, if $f\in\mathscr{G}\cap W$, since $W\subset F_0$, we have $f\in\mathscr{G}\cap F_0\cap W$, which is $\{0\}$. So the sum $\mathscr{G}+W$ is direct.

Step 1. — We prove that $\mathscr{F}_T$ is a closed subspace of $L^2(\mathbb{T})^d$ with finite codimension. For $f_0 \in \mathscr{G}$, the function $S(T; f_0, \mathscr{U} f_0)$ belongs to F^0, thus

$$\mathscr{K}(f_0) := -\mathrm{e}^{T\mathscr{L}^0} S(T; f_0, \mathscr{U} f_0) \tag{95}$$

is well-defined in F^0 by Proposition 19. Then, $\mathscr{K}$ is a compact operator on $\mathscr{G}$ because it has finite rank. By the Fredholm alternative, $(I+\mathscr{K})(\mathscr{G})$ is a closed subspace of $\mathscr{G}$ and there exists a closed subspace $\mathscr{G}'$ of $\mathscr{G}$, with finite codimension in $\mathscr{G}$, such that $(I+\mathscr{K})$ is a bijection from $\mathscr{G}'$ to $(I+\mathscr{K})(\mathscr{G})$. Note that $\mathscr{G}'$ is also a closed subspace with finite codimension in $L^2(\mathbb{T})^d$.

For any $f_0 \in \mathscr{G}'$, by using that $\mathscr{K}(f_0) \in F^0$ and (95), we obtain

$$S(T, \mathscr{K}(f_0), 0) = \mathrm{e}^{-T\mathscr{L}} \mathscr{K}(f_0) = \mathrm{e}^{-T\mathscr{L}^0} \mathscr{K}(f_0) = -S(T, f_0, \mathscr{U} f_0),$$

thus

$$S(T, f_0 + \mathscr{K}(f_0), \mathscr{U} f_0) = S(T, f_0, \mathscr{U} f_0) + S(T, \mathscr{K}(f_0), 0) = 0.$$

This proves that $\mathscr{F}_T$ contains $(I+\mathscr{K})(\mathscr{G}')$, which is a closed subspace with finite codimension in $L^2(\mathbb{T})^d$. Therefore, there exists a finite dimensional subspace $F_\sharp$ of $L^2(\mathbb{T})^d$ such that $\mathscr{F}_T = (I+\mathscr{K})(\mathscr{G}') \oplus F_\sharp$. This gives the conclusion of Step 1.

Step 2. — We prove that, up to a possibly smaller choice of $T > T^*$, there exists $\delta > 0$ such that $\mathscr{F}_{T'} = \mathscr{F}_T$ for every $T' \in [T, T+\delta]$. When $0 < T' < T''$, by extending controls defined on $(0, T')$ by zero on (T', T''), we see that $\mathscr{F}_{T'} \subset \mathscr{F}_{T''}$. Thus, the map $T' \mapsto \operatorname{codim}(\mathscr{F}_{T'})$ is decreasing and takes integer values. As a consequence the discontinuities on $(T^*, T+1]$ are isolated. If T is not such a discontinuity point, then there exists $\delta > 0$ such that $\operatorname{codim}(\mathscr{F}_{T'}) = \operatorname{codim}(\mathscr{F}_T)$ for every $T' \in [T, T+\delta]$. In case T is such a discontinuity point, one may replace T by a smaller value, still such that $T > T^*$, for which this holds.

Step 3. — We prove that $\left(\mathrm{e}^{-t\mathscr{L}^*} \mathscr{F}_T^\perp\right)^\perp \subset \mathscr{F}_T$ for every $t \in (0,\delta)$. Let $t \in (0,\delta)$ and $g_0 \in L^2(\mathbb{T})^d$ be such that $\langle g_0, \mathrm{e}^{-t\mathscr{L}^*} f_0\rangle = 0$ for every $f_0 \in \mathscr{F}_T^\perp$. Then $\langle \mathrm{e}^{-t\mathscr{L}} g_0, f_0\rangle = 0$ for every $f_0 \in \mathscr{F}_T^\perp$, i.e., $\mathrm{e}^{-t\mathscr{L}} g_0 \in (\mathscr{F}_T^\perp)^\perp$. By Step 1, $\mathscr{F}_T$ is a closed subspace of $L^2(\mathbb{T})^d$ thus $(\mathscr{F}_T^\perp)^\perp = \mathscr{F}_T$. Therefore $\mathrm{e}^{-t\mathscr{L}} g_0 \in \mathscr{F}_T$. By definition of $\mathscr{F}_T$, this implies that $g_0 \in \mathscr{F}_{T+t}$. By Step 2, we get $g_0 \in \mathscr{F}_T$, which ends the proof of Step 3.

Step 4. — We prove that $\mathscr{F}_T^\perp$ is left invariant by $\mathrm{e}^{-t\mathscr{L}^*}$, i.e., $\mathscr{F}_T^\perp = \mathrm{e}^{-t\mathscr{L}^*} \mathscr{F}_T^\perp$ for every $t > 0$. The subspace $\mathrm{e}^{-t\mathscr{L}^*} \mathscr{F}_T^\perp$ is closed in $L^2(\mathbb{T})^d$ because it has finite dimension. Thus $\left((\mathrm{e}^{-t\mathscr{L}^*} \mathscr{F}_T^\perp)^\perp\right)^\perp = \mathrm{e}^{-t\mathscr{L}^*} \mathscr{F}_T^\perp$ and we deduce from Step 3 that, for every $t \in (0,\delta)$, $\mathscr{F}_T^\perp \subset \mathrm{e}^{-t\mathscr{L}^*} \mathscr{F}_T^\perp$. Taking into account that $\dim(\mathrm{e}^{-t\mathscr{L}^*} \mathscr{F}_T^\perp) \leqslant \dim(\mathscr{F}_T^\perp)$, we obtain $\mathscr{F}_T^\perp = \mathrm{e}^{-t\mathscr{L}^*} \mathscr{F}_T^\perp$ for every $t \in (0,\delta)$. By the semi-group property, this equality holds for every $t > 0$.

Step 5. — We prove the existence of $N \in \mathbb{N}$ such that any $f_0 \in \mathscr{F}_T^\perp$ can be written

$$f_0 = \sum_{k \leqslant N} \varphi_k e_k \quad \text{with } \varphi_k \in \mathbb{C}^d. \tag{96}$$

Let $S(t)^*$ be the restriction of the semigroup $\mathrm{e}^{-t\mathscr{L}^*}$ to $\mathscr{F}_T^\perp$, i.e., $S(t)^* = \mathrm{e}^{-t\mathscr{L}^*}|_{\mathscr{F}_T^\perp}$. Then $S(t)^* = \mathrm{e}^{tM}$, where M is a matrix such that $\mathscr{L}^* f_0 = M f_0$ for every $f_0 \in \mathscr{F}_T^\perp$.

But then $\ker(M-\overline{\lambda})^j = \ker(\mathscr{L}^*-\overline{\lambda})^j \cap \mathscr{F}_T^\perp$. The Kernel decomposition theorem applied to M, and the structure of the generalized eigenspaces of $\mathscr{L}^*$ gives the conclusion of Step 4.

Step 6. — We prove that any element of $L^2(\mathbb{T})^d$ can be steered to $\mathscr{F}_T$ in an arbitrary short time, i.e., for every $\varepsilon > 0$ and $f_0 \in L^2(\mathbb{T})^d$, there exists

$$u \in L^2((0,T')\times\omega)^{d_1} \times C_c^\infty((T',T)\times\omega)^{d_2}$$

such that $S(\varepsilon; f_0, u) \in \mathscr{F}_T$. By the Hilbert Uniqueness Method, it is sufficient to prove an observability inequality for $S(t)^*$. By using the finite-dimensionality of $\mathscr{F}_T^\perp$, it is equivalent to prove that the following unique continuation property holds: if $f(t,\cdot) = \mathrm{e}^{tM} f_0$ with $f = 0$ in $(0,\varepsilon)\times\omega$, then $f_0 = 0$. By using the spectral inequality of Lebeau-Robbiano, i.e., (90) and (96), we readily get the result.

Step 7: Conclusion. — Step 5 implies the controllability of the system in any time $\tau > T$. As T is an arbitrary time such that $T > T^*$, this concludes the null-controllability in any time $T > T^*$. □

By a duality argument, we obtain the following result, that will be used in the next sections.

Corollary 32. — *For every $T > T^*$ and $s \in \mathbb{N}$, there exists $C_{T,s} > 0$ such that, for every $g_0 \in L^2(\mathbb{T})^d$ the solution $g(t) = \mathrm{e}^{-t\mathscr{L}^*} g_0 = (g_1, g_2)(t)$ of the adjoint system* (31) *satisfies*

$$\|g(T)\|_{L^2(\mathbb{T})^d} \leqslant C_{T,s}\big(\|g_1\|_{L^2(q_T)^{d_1}} + \|g_2\|_{H^{-s}(q_T)^{d_2}}\big),$$

where $q_T = (0,T)\times\omega$.

We will use the following standard lemma that gives a canonical isometry between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$.

Lemma 33. — *Let Ω be an open subset of $\mathbb{R}^d$ or a compact manifold (possibly with boundary). Let $s \geqslant 0$ and $\iota_s\colon H_0^s(\Omega) \to L^2(\Omega)$ be the inclusion map.*(9) *The map $\iota_s^*\colon L^2(\Omega) \to H_0^s(\Omega)$ extends to a bijective isometry from $H^{-s}(\Omega)$ to $H_0^s(\Omega)$.*

Proof. — The map ι_s^* is defined on $L^2(\Omega)$ by

$$\forall\, f \in L^2(\Omega),\ \forall\, v \in H_0^s(\Omega),\quad \langle \iota_s^* f, v\rangle_{H_0^s} = \langle f, v\rangle_{L^2}. \tag{97}$$

Thus, for every $f \in L^2(\Omega)$,

$$|\iota_s^* f|_{H_0^s} = \sup_{|v|_{H_0^s}=1} \langle \iota_s^* f, v\rangle_{H_0^s} = \sup_{|v|_{H_0^s}=1} \langle f, v\rangle_{L^2} = |f|_{H^{-s}},$$

where we used the definition of $H^{-s}(\Omega)$ as the dual of $H_0^s(\Omega)$ with respect to the pivot space $L^2(\Omega)$ (see for instance [27, §2.9]). Since $L^2(\Omega)$ is dense in $H^{-s}(\Omega)$, this proves that ι_s^* extends by continuity to $H^{-s}(\Omega)$.

(9)We recall that $H_0^s(\Omega)$ is the closure of $C_c^\infty(\Omega)$ for the H^s-norm, and that $H^{-s}(\Omega)$ is the dual of $H_0^s(\Omega)$ with respect to the pivot space $L^2(\Omega)$.

This extension is an isometry from $H^{-s}(\Omega)$ onto its range. As such it is injective and its range is closed. To prove it is bijective, we check that its range is dense, i.e., that its orthogonal is zero.

If $g_0 \in H_0^s(\Omega)$ is orthogonal to $\operatorname{Im}(\iota_s^*)$, then, according to the definition of ι_s^* (Equation (97)) g_0 is orthogonal in $L^2(\Omega)$ to $H_0^s(\Omega)$. But $H_0^s(\Omega)$ is dense in $L^2(\Omega)$, so $g_0 = 0$. Thus $\operatorname{Im}(\iota_s^*)^\perp = \{0\}$. □

Proof of Corollary 32. — We apply the duality lemma 14 with

$$\Phi_2\colon f_0 \in L^2(\mathbb{T})^d \longmapsto f(T,\cdot) \in L^2(\mathbb{T})^d,$$

where f is the solution to the system (Sys) with initial data f_0 and control $u = 0$, and

$$\Phi_3\colon u = (u_1, u_2) \in L^2(q_T)^{d_1} \times H_0^s(q_T)^{d_2} \longmapsto f(T,\cdot) \in L^2(\mathbb{T})^d,$$

where f is the solution to the system (Sys) with initial data $f_0 = 0$ and control u. The null-controllability result proved above is equivalent to the inclusion $\operatorname{Im}(\Phi_2) \subset \operatorname{Im}(\Phi_3)$, thus to the existence of $C > 0$ such that for every $g_T \in L^2(\mathbb{T})^d$,

$$\|\Phi_2^*(g_T)\|_{L^2(\mathbb{T})^d} \leqslant C\|\Phi_3^*(g_T)\|_{L^2(q_T)^{d_1}\times H_0^s(q_T)^{d_2}}. \tag{98}$$

We compute the adjoint operators of Φ_2 and Φ_3 thanks to the duality relation between the solution f of (Sys) and the solution $\varphi(\cdot) = g(T-\cdot)$ of the adjoint system (31):

$$\begin{aligned}\langle f(T), \varphi(T)\rangle_{L^2(\mathbb{T})^d} &= \langle f(0), \varphi(0)\rangle_{L^2(\mathbb{T})^d} + \int_0^T \int_\omega \langle u(t,x), \varphi(t,x)\rangle \,\mathrm{d}t\,\mathrm{d}x \\ &= \langle f(0), \varphi(0)\rangle_{L^2(\mathbb{T})^d} + \int_0^T \int_\omega \langle u_1(t,x), \varphi_1(t,x)\rangle + \langle u_2(t,x), \varphi_2(t,x)\rangle \,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{99}$$

First, we have $\Phi_2^*(g_T) = (\mathrm{e}^{-T\mathscr{L}})^* g_T = \mathrm{e}^{-T\mathscr{L}^*} g_T$. To compute Φ_3^*, we introduce the input-output operator $\mathscr{F}_T\colon u \in L^2(q_T)^d \mapsto f(T,.) \in L^2(\mathbb{T})^d$, where f is the solution of (Sys) with initial condition $f_0 = 0$ and right-hand side u. By (99), $\mathscr{F}_T^*(g_T)$ is the restriction of $\mathrm{e}^{(t-T)\mathscr{L}^*} g_T$ to $[0,T]\times\omega$. We have $\Phi_3 = \mathscr{F}_T \circ (I, \iota_s)$, where (I, ι_s) stands for the inclusion map $L^2(q_T)^{d_1} \times H_0^s(q_T)^{d_2} \to L^2(q_T)^d$. Thus, according to Lemma 33, the right-hand side of the inequality (98) is

$$\|(I, \iota_s^*) \circ \mathscr{F}_T^*(g_T)\|_{L^2(q_T)^{d_1}\times H_0^s(q_T)^{d_2}} = \|\mathscr{F}_T^*(g_T)\|_{L^2(q_T)^{d_1}\times H^{-s}(q_T)^{d_2}},$$

which gives the conclusion. □

5. Hyperbolic control: coupling of order zero

The goal of this section is to prove the following result on the system

$$\begin{cases}(\partial_t + A'\partial_x + K_{11})f_1 + (A_{12}\partial_x + K_{12})f_2 = u_1 1_\omega & \text{in } (0,T)\times\mathbb{T},\\ \big(\partial_t - \partial_x^2 + K_{22}\big)f_2 + K_{21}f_1 = 0 & \text{in } (0,T)\times\mathbb{T}.\end{cases} \tag{100}$$

Theorem 34. — *We assume* (H.1)–(H.4), $D = I_{d_2}$ $m = d_1$, $M_1 = I_{d_1}$, $M_2 = 0$, $A_{21} = 0$ *and* $A_{22} = 0$. *Let* T^* *be defined by* (3). *The following statements are equivalent.*

– *The system* (100) *is null-controllable in any time* $T > T^*$.

– *The pair of matrices* (K_{22}, K_{21}) *satisfies the Kalman rank condition:*

$$\operatorname{Span}\{K_{22}^j K_{21} X_1; X_1 \in \mathbb{C}^{d_1}, 0 \leqslant j \leqslant d_2 - 1\} = \mathbb{C}^{d_2}. \tag{101}$$

The interest of this theorem is that its proof is essentially the same as that of Theorems 3 and 4 (that will be done in the next sections) but it is less technical.

In Section 5.1, we prove that the Kalman condition (101) is necessary for the null-controllability of the system (100). In Section 5.2, we prove that the Kalman condition (101) is sufficient for the null-controllability of the system (100), first in the case $d_1 = 1$ (i.e., with one hyperbolic line in the system) where the cascade structure (or Brunovski form) is easy to handle, then in the general case $d_1 > 1$ which is more delicate to write.

5.1. The Kalman condition is necessary. — If the null-controllability property for (100) holds, then, by considering the Fourier components of the solution and the control, we obtain the null-controllability, for every $n \in \mathbb{Z} \smallsetminus \{0\}$, of the system

$$\begin{cases} X_1(t)' + (\mathrm{i}nA' + K_{11})X_1(t) + (\mathrm{i}nA_{12} + K_{12})X_1(t) = v_1(t), \\ X_2'(t) + (n^2 I_{d_2} + K_{22})X_2(t) + K_{21}X_1(t) = 0, \end{cases}$$

with state $X(t) = (X_1, X_2)(t) \in \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$ and control $v_1 \in L^2(0,T)^{d_1}$. This requires the null-controllability of the control system

$$X_2'(t) + (n^2 I_{d_2} + K_{22})X_2(t) + K_{21}X_1(t) = 0,$$

with state $X_2(t) \in \mathbb{C}^{d_2}$ and control $X_1 \in L^2(0,T)^{d_1}$, i.e., the Kalman rank condition (see for instance [12, Th. 1.16])

$$\operatorname{Span}\{(n^2 I_{d_2} + K_{22})^j K_{21} v_1; v_1 \in \mathbb{C}^{d_1},\ j \in \{0, \dots, d_2 - 1\}\} = \mathbb{C}^{d_2},$$

that can equivalently be written in the form (101).

5.2. The Kalman condition is sufficient. — In this section, we explain how to complete the proof of Theorem 2 to prove that the Kalman rank condition (101) implies the null-controllability of (100) in time $T > T^*$, in Theorem 34.

First, we treat the case $d_1 = 1$, then we generalize to the case $d_1 > 1$. From now and until end of this subsection, C will denote positive constants which will vary from line to line. For $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant d_i$, we denote by v_i^j the j-th component of a vector $v_i \in \mathbb{C}^{d_i}$.

5.2.1. *The case of one hyperbolic component:* $d_1 = 1$. — By using Hamilton-Cayley's theorem, we know that there exist $c_0, \dots, c_{d_2-1} \in \mathbb{R}$ such that

$$K_{22}^{d_2} = c_0 I_{d_2} + c_1 K_{22} + \dots + c_{d_2-1} K_{22}^{d_2-1}. \tag{102}$$

By using the Kalman condition (101), the matrix P defined as follows

$$P := (K_{21}, K_{22}K_{21}, \dots, K_{22}^{d_2-1}K_{21}), \tag{103}$$

is invertible. We set

$$\widehat{K_{22}} := \begin{pmatrix} 0 & \dots & \dots & 0 & c_0 \\ 1 & 0 & \dots & \vdots & c_1 \\ 0 & \ddots & \ddots & \vdots & c_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & c_{d_2-1} \end{pmatrix} \quad \text{and} \quad \widehat{K_{21}} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{104}$$

From (102), (103), (104), we check that we have the following relations

$$K_{22}P = P\widehat{K_{22}} \text{ and } K_{21} = P\widehat{K_{21}}, \quad \text{i.e., } \widehat{K_{22}} = P^{-1}K_{22}P \text{ and } \widehat{K}_{21} = P^{-1}K_{21}.$$

The function $w = (w_1, w_2) = (f_1, P^{-1}f_2)$ solves

$$\begin{cases} (\partial_t + A'\partial_x + K_{11})w_1 + (A_{12}P\partial_x + K_{12}P)w_2 = u_1 1_\omega & \text{in } (0,T)\times\mathbb{T}, \\ \Big(\partial_t - \partial_x^2 + \widehat{K_{22}}\Big)w_2 + \widehat{K_{21}}w_1 = 0 & \text{in } (0,T)\times\mathbb{T}, \\ (w_1, w_2)(0,\cdot) = (w_{01}, w_{02}) & \text{in } \mathbb{T}. \end{cases} \tag{105}$$

The system (105) is a "cascade system". Indeed, roughly speaking the control u_1 directly controls the component w_1, the component w_1 indirectly controls the component w_2^1 in the second equation through the coupling term w_1, the component w_2^1 indirectly controls the component w_2^2 in the third equation through the coupling term $w_2^1, \dots$ the component $w_2^{d_2-1}$ indirectly controls the component $w_2^{d_2}$ in the last equation through the coupling term $w_2^{d_2-1}$.

The adjoint system of (105) is

$$\begin{cases} \big(\partial_t - A'^{\mathrm{tr}}\partial_x + K_{11}^{\mathrm{tr}}\big)g_1 + \widehat{K_{21}}^{\mathrm{tr}}g_2 = 0 & \text{in } (0,T)\times\mathbb{T}, \\ \big(\partial_t - \partial_x^2 + \widehat{K_{22}}^{\mathrm{tr}}\big)g_2 + \big(-(A_{12}P)^{\mathrm{tr}}\partial_x + (K_{12}P)^{\mathrm{tr}}\big)g_1 = 0 & \text{in } (0,T)\times\mathbb{T}, \\ (g_1, g_2)(0,\cdot) = (g_{01}, g_{02}) & \text{in } \mathbb{T}. \end{cases} \tag{106}$$

From Corollary 32, we know that for every $g_0 \in L^2(\mathbb{T})^d$, the solution g of (106) satisfies

$$\|g(T,\cdot)\|_{L^2(\mathbb{T})} \leqslant C\big(\|g_1\|_{L^2(q_T)} + \|g_2\|_{H^{-2d_2+1}(q_T)}\big). \tag{107}$$

By using the fact that $\widehat{K}_{22}$ is a companion matrix, see (104), for every $i \in \{2, \dots, d_2\}$, the i-th equation of (106) is

$$\partial_t g_2^{i-1} - \partial_x^2 g_2^{i-1} + g_2^i + b_{i-1}\partial_x g_1 + a_{i-1}g_1 = 0, \quad \text{with } (a_{i-1}, b_{i-1}) \in \mathbb{R}^2$$

Then we deduce

$$\|g_2^i\|_{H^{-2i+1}(q_T)} \leqslant C\big(\|g_2^{i-1}\|_{H^{-2(i-1)+1}(q_T)} + \|g_1\|_{L^2(q_T)}\big). \tag{108}$$

Here, we have used in particular that

$$\|(\partial_t - \partial_x^2)g_2^{i-1}\|_{H^{-2i+1}(q_T)} \leqslant C\|g_2^{i-1}\|_{H^{-2(i-1)+1}(q_T)}$$

and

$$\|b_{i-1}\partial_x g_1 + a_{i-1}g_1\|_{H^{-2i+1}(q_T)} \leqslant C\|g_1\|_{L^2(q_T)}.$$

We deduce from (107) and (108) that

$$\|g(T,\cdot)\|_{L^2(\mathbb{T})} \leqslant C\big(\|g_1\|_{L^2(q_T)} + \|g_2^1\|_{H^{-1}(q_T)}\big). \tag{109}$$

By using the fact that $\widehat{K}_{21}$ is the first vector of the canonical basis of $\mathbb{R}^{d_2}$, see (104), the first equation of (106) is

$$\partial_t g_1 - A'\partial_x g_1 + K_{11}g_1 + g_2^1 = 0.$$

We obtain

$$\|g_2^1\|_{H^{-1}(q_T)} \leqslant C\|g_1\|_{L^2(q_T)}. \tag{110}$$

So, we deduce from (109) and (110) the observability inequality

$$\|g(T,\cdot)\|^2_{L^2(\mathbb{T})^d} \leqslant C\int_0^T\int_\omega |g_1(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t,$$

in the case $d_1 = 1$. This concludes the proof of Theorem 34 in the case $d_1 = 1$ by duality.

5.2.2. *The case of several hyperbolic components:* $d_1 > 1$. — In this section, we deal with the general problem of null-controllability of (100). To this end, we introduce $K_{21}^i \in \mathbb{R}^{d_2}$ the i-th column of the matrix K_{21} ($1 \leqslant i \leqslant d_1$), i.e.,

$$K_{21} = \big(K_{21}^1|K_{21}^2|\dots|K_{21}^{d_1}\big),$$

From the Kalman rank condition (101), we construct an adapted basis of $\mathbb{C}^{d_2}$.

Lemma 35. — *There exist $r \in \{1,\dots,d_2\}$ and sequences $(\ell_j)_{1\leqslant j\leqslant r} \subset \{1,2,\dots,d_1\}$ and $(s_j)_{1\leqslant j\leqslant r} \subset \{1,2,\dots,d_2\}$ with $\sum_{j=1}^r s_j = d_2$, such that*

$$\mathscr{B} = \bigcup_{j=1}^r \big\{K_{21}^{\ell_j}, K_{22}K_{21}^{\ell_j},\dots,K_{22}^{s_j-1}K_{21}^{\ell_j}\big\}$$

is a basis of $\mathbb{C}^{d_2}$. Moreover, for every j, with $1 \leqslant j \leqslant r$, there exist $\alpha^i_{k,s_j} \in \mathbb{R}$ ($1 \leqslant i \leqslant j$, $1 \leqslant k \leqslant s_j$) such that

$$K_{22}^{s_j}K_{21}^{\ell_j} = \sum_{i=1}^j \big(\alpha^i_{1,s_j}K_{21}^{\ell_i} + \alpha^i_{2,s_j}K_{22}K_{21}^{\ell_i} + \dots + \alpha^i_{s_i,s_j}K_{22}^{s_i-1}K_{21}^{\ell_i}\big). \tag{111}$$

For a proof of this lemma, see [3, Lem. 3.1]. Let $\mathscr{B}$ the basis of $\mathbb{C}^{d_2}$ provided by Lemma 35 and P be the matrix whose columns are the elements of $\mathscr{B}$, i.e.,

$$P := \Big(K_{21}^{\ell_1}|K_{22}K_{21}^{\ell_1}|\dots|K_{22}^{s_1-1}K_{21}^{\ell_1}|\dots|K_{21}^{\ell_r}|\dots|K_{22}^{s_r-1}K_{21}^{\ell_r}\Big).$$

Let us observe that the basis $\mathscr{B}$ has been constructed in such a way that (111) is satisfied.

Let the matrices $C_{ii} \in \mathbb{R}^{s_i \times s_i}$ and $C_{ij} \in \mathbb{R}^{s_i \times s_j}$, $1 \leqslant i < j \leqslant r$, be defined by

$$C_{ii} = \begin{pmatrix} 0 & 0 & 0 & \dots & \alpha^i_{1,s_i} \\ 1 & 0 & 0 & \dots & \alpha^i_{2,s_i} \\ 0 & 1 & 0 & \dots & \alpha^i_{3,s_i} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \alpha^i_{s_i,s_i} \end{pmatrix} \quad \text{and} \quad C_{ij} = \begin{pmatrix} 0 & \dots & 0 & \alpha^i_{1,s_j} \\ \vdots & \ddots & \vdots & \alpha^i_{2,s_j} \\ 0 & \dots & 0 & \alpha^i_{s_i,s_j} \end{pmatrix}. \tag{112}$$

We set

$$\widehat{K_{22}} := \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1r} \\ 0 & C_{22} & \dots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{rr} \end{pmatrix} \quad \text{and} \quad \widehat{K_{21}} := P^{-1} K_{21}. \tag{113}$$

From (111), (113) and (112), by denoting $P_i := \Big(K_{21}^{\ell_i} | K_{22} K_{21}^{\ell_i} | \dots | K_{22}^{s_i - 1} K_{21}^{\ell_i}\Big)$, we obtain

$$\begin{aligned} K_{22}\, P_i &= \big(K_{22} K_{21}^{\ell_i} | K_{22}^2 K_{21}^{\ell_i} | \dots | K_{22}^{s_i} K_{21}^{\ell_i}\big) \\ &= \Big(K_{22} K_{21}^{\ell_i} \big| K_{22}^2 K_{21}^{\ell_i} \big| \dots \big| \textstyle\sum_{k=1}^{i} \big(\alpha^k_{1,s_i} K_{21}^{\ell_k} + \alpha^k_{2,s_i} K_{22} K_{21}^{\ell_k} + \dots + \alpha^k_{s_k,s_i} K_{22}^{s_k - 1} K_{21}^{\ell_k}\big)\Big) \\ &= \Big(0 \big| \dots \big| 0 \big| \textstyle\sum_{k=1}^{i-1} \big(\alpha^k_{1,s_i} K_{21}^{\ell_k} + \alpha^k_{2,s_i} K_{22} K_{21}^{\ell_k} + \dots + \alpha^k_{s_k,s_i} K_{22}^{s_k - 1} K_{21}^{\ell_k}\big)\Big) \\ &\quad + \big(K_{22} K_{21}^{\ell_i} \big| K_{22}^2 K_{21}^{\ell_i} \big| \dots \big| \big(\alpha^i_{1,s_i} K_{21}^{\ell_i} + \alpha^i_{2,s_i} K_{22} K_{21}^{\ell_i} + \dots + \alpha^i_{s_i,s_i} K_{22}^{s_i - 1} K_{21}^{\ell_i}\big)\big) \\ &= P_1 C_{1i} + P_2 C_{2i} + \dots + P_i C_{ii}. \end{aligned}$$

Therefore,

$$K_{22} P = P \widehat{K_{22}} \quad \text{and} \quad P e_{S_i} = K_{21}^{\ell_i},\ 1 \leqslant i \leqslant r, \tag{114}$$

where e_{S_i} is the S_i-element of the canonical basis of $\mathbb{C}^{d_2}$ with $S_i = 1 + \sum_{j=1}^{i-1} s_j$. In the following, we will also use the notation $S_{r+1} := d_2 + 1$.

We argue as in the previous subsection. We perform the same change of variable $w = (w_1, w_2) = (f_1, P^{-1} f_2)$, we consider the solution g of the adjoint system

$$\begin{cases} (\partial_t - A'^{\mathrm{tr}} \partial_x + K_{11}^{\mathrm{tr}}) g_1 + \widehat{K_{21}}^{\mathrm{tr}} g_2 = 0 & \text{in } (0,T) \times \mathbb{T}, \\ \Big(\partial_t - \partial_x^2 + \widehat{K_{22}}^{\mathrm{tr}}\Big) g_2 + (-(A_{12} P)^{\mathrm{tr}} \partial_x + (K_{12} P)^{\mathrm{tr}}) g_1 = 0 & \text{in } (0,T) \times \mathbb{T}, \\ (g_1, g_2)(0, \cdot) = (g_{01}, g_{02}) & \text{in } \mathbb{T}. \end{cases} \tag{115}$$

We recall from Corollary 32 that the solution g of (115) satisfies

$$\|g(T,\cdot)\|_{L^2(\mathbb{T})} \leqslant C\big(\|g_1\|_{L^2(q_T)} + \|g_2\|_{H^{-2m+1}(q_T)}\big), \text{ with } m = \max_{1 \leqslant i \leqslant r} s_i. \tag{116}$$

We use the coupling terms in the system (115) in order to get rid of the term $\|g_2\|^2_{H^{-2m+1}(q_T)}$ in the right-hand side of the inequality (116).

From the cascade form of the matrix $\widehat{K_{22}}$, see (113), more precisely from the cascade form of the block matrix C_{ii} and the form of the matrices $C_{1,i}, \dots, C_{i-1,i}$, see (112), the equations of the adjoint system (115) are

(117) $\quad \forall\, i \in \{1,\dots,r\},\ \forall\, j \in \{S_i,\dots,S_{i+1}-2\}$

$$\partial_t g_2^j - \partial_x^2 g_2^j + g_2^{j+1} + \sum_{k=1}^{d_1} b_{i,j}^k \partial_x g_1^k + a_{i,j}^k g_1^k = 0, \qquad (a_{i,j}^k, b_{i,j}^k) \in \mathbb{R}^2.$$

We deduce successively from (117) with

$$j = S_{i+1}-2, S_{i+1}-3, \dots, S_{i+1}-2-(s_i-2) = S_i,$$

the following estimates

$$\begin{aligned}\big\|g_2^{S_{i+1}-1}\big\|_{H^{-2s_i+1}(q_T)} &\leqslant C\big(\big\|g_2^{S_{i+1}-2}\big\|_{H^{-2(s_i-1)+1}(q_T)} + \|g_1\|_{L^2(q_T)}\big)\\ &\leqslant C\big(\big\|g_2^{S_{i+1}-3}\big\|_{H^{-2(s_i-2)+1}(q_T)} + \|g_1\|_{L^2(q_T)}\big)\\ &\leqslant \cdots\\ &\leqslant C\big(\big\|g_2^{S_{i+1}-2-(s_i-2)}\big\|_{H^{-1}(q_T)} + \|g_1\|_{L^2(q_T)}\big).\end{aligned}$$

So, we have for every $i \in \{1,\dots,r\}$ and $j \in \{S_i+1,\dots,S_{i+1}-1\}$,

$$\big\|g_2^j\big\|_{H^{-2m+1}(q_T)} \leqslant C\big(\big\|g_2^{S_i}\big\|_{H^{-1}(q_T)} + \|g_1\|_{L^2(q_T)}\big). \tag{118}$$

Then, by using (113) and (114), we have $\widehat{K_{21}}^{\ell_i} = P^{-1}K_{21}^{\ell_i} = e_{S_i}$. Consequently, the ℓ_i-th equation of the adjoint system (115) is

$$\partial_t g_1^{\ell_i} + \sum_{k=1}^{d_1} a_{\ell_i,k}\partial_x g_1^k + b_{\ell_i,k} g_1^k + g_2^{S_i} = 0, \qquad (a_{\ell_i,k}, b_{\ell_i,k}) \in \mathbb{R}^2.$$

Then, we obtain

$$\big\|g_2^{S_i}\big\|_{H^{-1}(q_T)} \leqslant C\|g_1\|_{L^2(q_T)}. \tag{119}$$

By gathering (118) and (119), we obtain

(120) $\quad \forall\, i \in \{1,\dots,r\},\ \forall\, j \in \{S_i,\dots,S_{i+1}-1\},\ \big\|g_2^j\big\|_{H^{-2m+1}(q_T)} \leqslant C\|g_1\|_{L^2(q_T)}.$

By using that $\{S_1,\dots,S_2-1,S_2,\dots,S_3-1,\dots,S_r,\dots,S_{r+1}-1\} = \{1,\dots,d_2\}$, we finally deduce from (120) and (116) the observability inequality

$$\|g(T,\cdot)\|_{L^2(\mathbb{T})^d}^2 \leqslant C\int_0^T\int_\omega |g_1(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

This concludes the proof of Theorem 34 in the case $d_1 > 1$ by duality. $\square$

6. Hyperbolic control: coupling of order one

The goal of this section is to prove Theorem 3. The requirement of the Kalman rank condition (6) for null-controllability is an adaptation of the proof given in Section 5.1. Now, we explain how to complete the proof of Theorem 2 to prove that the Kalman condition is sufficient for null-controllability. We set

$$\mathbf{F_2} := L^2(\mathbb{T})^{d_1} \times L^2_{\mathrm{m}}(\mathbb{T})^{d_2} = \Big\{f_0 = (f_{01}, f_{02}) \in L^2(\mathbb{T})^d\ ;\ \int_{\mathbb{T}} f_{02}(x)\,\mathrm{d}x = 0\Big\}. \tag{121}$$

We only give the proof in the case $d_1 = 1$. The case $d_1 > 1$ is an easy adaptation of the case $d_1 = 1$ and the arguments already presented for coupling terms of order zero in Section 5.2.2.

6.1. A special observability inequality. — The goal of this section is to prove the following observability inequality.

Proposition 36. — *There exists $C > 0$ such that for every $g_0 \in \mathbf{F_2}$, the solution of the adjoint system* (31) *satisfies*

$$\|g(T,\cdot)\|^2_{L^2(\mathbb{T})} \leqslant C\big(\|g_1\|^2_{L^2(q_T)} + \|\partial_x^{d_2} g_2\|^2_{H^{-2d_2+1}(q_T)}\big). \tag{122}$$

In order to prove Proposition 36, by a duality argument, it is sufficient to establish the following null-controllability result.

Proposition 37. — *For every $f_0 \in \mathbf{F_2}$, there exists $u \in L^2(q_T)^{d_1} \times (H_0^{2d_2-1}(q_T))^{d_2}$ such that $S(T, f_0, (u_{\mathrm{h}}, \partial_x^{d_2} u_{\mathrm{p}})) = 0$.*

Proof of the equivalence between Propositions 36 *and* 37. — We apply Lemma 14 with

$$\Phi_2 \colon f_0 \in \mathbf{F_2} \longmapsto f(T,\cdot) \in \mathbf{F_2},$$

where f is the solution to the system (Sys) with initial data f_0 and control $u = 0$, and

$$\Phi_3 \colon u = (u_1, u_2) \in L^2(q_T)^{d_1} \times H_0^{2d_2-1}(q_T)^{d_2} \longmapsto f(T,\cdot) \in \mathbf{F_2},$$

where f is the solution to the system (Sys) with initial data $f_0 = 0$ and control $(u_1, \partial_x^{d_2} u_2)$. Note that by integrating the second equation of the system (1), we see that a control of the form $(u_1, \partial_x^{d_2} u_2)$ cannot change the mean of the parabolic component. This justifies that Φ_2 and Φ_3 do indeed take values in $\mathbf{F_2}$.

The null-controllability result of Proposition 37 is equivalent to the existence of $C > 0$ such that for every $g_T \in L^2(\mathbb{T})^d$,

$$\|\Phi_2^*(g_T)\|_{\mathbf{F_2}} \leqslant C\|\Phi_3^*(g_T)\|_{L^2(q_T)^{d_1} \times H_0^{2d_2-1}(q_T)^{d_2}}. \tag{123}$$

We have $\Phi_2^*(g_T) = (\mathrm{e}^{-T\mathscr{L}})^* g_T = \mathrm{e}^{-T\mathscr{L}^*} g_T$. We claim that the right-hand side of the inequality (123) satisfies

$$\|\Phi_3^*(g_T)\|_{L^2(q_T)^{d_1} \times H_0^{2d_2-1}(q_T)^{d_2}} = \|(g_1, (-1)^{d_2} \partial_x^{d_2} g_2)\|_{L^2(q_T)^{d_1} \times H^{-2d_2+1}(q_T)^{d_2}}, \tag{124}$$

where $g = \mathrm{e}^{-(T-t)\mathscr{L}^*} g_T$. This will prove that the inequality (123) is exactly the observability inequality (122).

We write Φ_3 as

$$\Phi_3 = \mathscr{F}_T \circ (I, \partial_x^{d_2}) \circ (I, \iota_{2d_2-1}),$$

where $\mathscr{F}_T \colon L^2(q_T)^d \to L^2(\mathbb{T})^d$ is the input-output operator introduced in the proof of Corollary 32, $\partial_x^{d_2}$ is seen as an unbounded operator on $L^2(\mathbb{T})^{d_2}$ with domain $H^{d_2}(\mathbb{T})^{d_2}$, and $\iota_{2d_2-1} \colon H^{2d_2-1}(q_T)^{d_2} \to L^2(q_T)^{d_2}$ is the inclusion map (see Lemma 33). Note that while Φ_3 written this way looks like an unbounded operator (because $\partial_x^{d_2}$ is), we have $\operatorname{Im}(\iota_{2d_2-1}) \subset D(\partial_x^{d_2})$, so that the composition of operators above is indeed a continuous operator. So, we have

$$\Phi_3^* = (I, \iota_{2d_2-1}^*) \circ (I, (\partial_x^{d_2})^*) \circ \mathscr{F}_T^* = (I, \iota_{2d_2-1}^*) \circ (I, (-1)^{d_2} \partial_x^{d_2}) \circ \mathscr{F}_T^*.$$

Since $\iota_{2d_2-1}^*$ is an isometry between $H_0^{2d_2-1}$ and H^{-2d_2+1} (see Lemma 33), this proves the relation (124). $\square$

First, we show that the null-controllability result of Proposition 37 is true at the high-frequency level, i.e., we prove the following adaptation of Proposition 20.

Proposition 38. — *There exist a closed subspace $\mathscr{G}^\blacklozenge$ of $L^2(\mathbb{T})^d$ with finite codimension and a continuous operator*

$$\begin{aligned}\mathscr{U}^\blacklozenge : \mathscr{G}^\blacklozenge &\longrightarrow L^2((0,T')\times\omega)^{d_1} \times C_c^\infty((T',T)\times\omega)^{d_2}\\ f_0 &\longmapsto (u_{\mathrm{h}}, u_{\mathrm{p}}),\end{aligned}$$

that associates with each $f_0 \in \mathscr{G}^\blacklozenge$ a pair of controls $\mathscr{U}^\blacklozenge f_0 = (u_{\mathrm{h}}, u_{\mathrm{p}})$ such that

$$\forall\, f_0 \in \mathscr{G}^\blacklozenge,\ \Pi S(T; f_0, (u_{\mathrm{h}}, \partial_x^{d_2} u_{\mathrm{p}})) = 0. \tag{125}$$

In order to prove Proposition 38, it is enough to prove Proposition 22 with parabolic control of the form $\partial_x^{d_2} u_{\mathrm{p}}$. Thus, by using Section 4.4.1, it is sufficient to show the following adaptation of Proposition 25.

Proposition 39. — *If n_0 is large enough, then for every $T > 0$, there exists a continuous operator*

$$\begin{aligned}\underline{\mathscr{U}}_T^{\mathrm{p};\blacklozenge} : F^{\mathrm{p}} &\longrightarrow C_c^\infty((0,T)\times\omega)^{d_2}\\ f_0 &\longmapsto u_{\mathrm{p}},\end{aligned}$$

that associates with each $f_0 \in F^{\mathrm{p}}$ a control $\underline{\mathscr{U}}_T^{\mathrm{p};\blacklozenge} f_0 = u_{\mathrm{p}}$ such that

$$\Pi^{\mathrm{p}} S(T; f_0, (0, \partial_x^{d_2} u_{\mathrm{p}})) = 0.$$

Proof. — Let $f_0 \in F^{\mathrm{p}}$ and $f_0^\blacklozenge$ be such that $\partial_x^{d_2} f_0^\blacklozenge = f_0$. Note that $f_0^\blacklozenge$ is well-defined because $\int_{\mathbb{T}} f_0(x)dx = 0$. We know from Proposition 25 that there exists $u_{\mathrm{p}} \in C_c^\infty((0,T)\times\omega)^{d_2}$ such that the solution $f^\blacklozenge$ of (Sys) with initial data $f_0^\blacklozenge$ and control $(0, u_{\mathrm{p}})$ satisfies

$$\Pi^{\mathrm{p}} f^\blacklozenge(T,\cdot) = 0.$$

Then, by setting $f := \partial_x^{d_2} f^\blacklozenge$ and by applying $\partial_x^{d_2}$ to the system (Sys) satisfied by $f^\blacklozenge$, we deduce that f is the solution of (Sys) with initial data f_0 and control $(0, \partial_x^{d_2} u_{\mathrm{p}})$ and satisfies

$$\Pi^{\mathrm{p}} f(T,\cdot) = 0,$$

because $\partial_x^{d_2}$ and Π^{p} commute.

We get the conclusion of the proof of Proposition 39 with the continuous operator $\underline{\mathscr{U}}_T^{\mathrm{p};\blacklozenge}(f_0) = \underline{\mathscr{U}}_T^{\mathrm{p}}(f_0^\blacklozenge)$ where $\underline{\mathscr{U}}_T^{\mathrm{p}}$ is the operator defined in Proposition 25. $\square$

Secondly, we have to show that the null-controllability result of Proposition 37 is true at the low frequency-level, as we have already shown for Theorem 2 in Section 4.5. All the steps of Section 4.5 remain unchanged except Step 6. Indeed, the unique continuation argument transforms into: if $f(t,\cdot) = \mathrm{e}^{tM} f_0$ with $(f_1, \partial_x^{d_2} f_2) = (0,0)$ in $(0,\varepsilon)\times\omega$ then $(f_{01}, \partial_x^{d_2} f_{02}) = (0,0)$ thanks to the spectral inequality of Lebeau-Robbiano (90), that is to say, $f_0 = 0$ because $\int_{\mathbb{T}} f_{02}(x)\,\mathrm{d}x = 0$.

This concludes the proof of Proposition 37, thus the proof of Proposition 36. $\square$

6.2. The case of one hyperbolic component: $d_1 = 1$. — Let us now give the proof of Theorem 3.

By Hamilton-Cayley's theorem, there exist $c_0, \dots, c_{d_2-1} \in \mathbb{R}$ such that

$$A_{22}^{d_2} = c_0 I_{d_2} + c_1 A_{22} + \cdots + c_{d_2-1} A_{22}^{d_2-1}.$$

By using the Kalman condition (6), the matrix P defined as follows

$$P := (A_{21}, A_{22}A_{21}, \dots, A_{22}^{d_2-1}A_{21}),$$

is invertible. By setting

$$\widehat{A_{22}} := \begin{pmatrix} 0 & \dots & \dots & 0 & c_0 \\ 1 & 0 & \dots & \vdots & c_1 \\ 0 & \ddots & \ddots & \vdots & c_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & c_{d_2-1} \end{pmatrix} \quad \text{and} \quad \widehat{A_{21}} := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{126}$$

we check that we have the following relations

$$A_{22}P = P\widehat{A_{22}} \text{ and } A_{21} = P\widehat{A_{21}}, \text{ i.e., } \widehat{A_{22}} = P^{-1}A_{22}P \text{ and } \widehat{A_{21}} = P^{-1}A_{21}.$$

Then, by setting $w = (w_1, w_2) = (f_1, P^{-1}f_2)$, we have

$$\begin{cases} (\partial_t + A'\partial_x + K_{11})w_1 + (A_{12}P\partial_x + K_{12}P)w_2 = u_1 1_\omega & \text{in } (0,T)\times\mathbb{T}, \\ \Big(\partial_t - \partial_x^2 + \widehat{A_{22}}\partial_x\Big)w_2 + \widehat{A_{21}}\partial_x w_1 = 0 & \text{in } (0,T)\times\mathbb{T}, \\ (w_1, w_2)(0,\cdot) = (w_{01}, w_{02}) & \text{in } \mathbb{T}. \end{cases} \tag{127}$$

The system (127) is a "cascade system" with coupling terms of order one in the spatial variable.

The adjoint system of (127) is

$$\begin{cases} \big(\partial_t - A'^{\mathrm{tr}}\partial_x + K_{11}{}^{\mathrm{tr}}\big)g_1 - \widehat{A_{21}}{}^{\mathrm{tr}}\partial_x g_2 = 0 & \text{in } (0,T)\times\mathbb{T}, \\ \big(\partial_t - \partial_x^2 - \widehat{A_{22}}{}^{\mathrm{tr}}\partial_x\big)g_2 + \big(-(A_{12}P)^{\mathrm{tr}}\partial_x + (K_{12}P)^{\mathrm{tr}}\big)g_1 = 0 & \text{in } (0,T)\times\mathbb{T}, \\ (g_1, g_2)(0,\cdot) = (g_{01}, g_{02}) & \text{in } \mathbb{T}. \end{cases} \tag{128}$$

We know from Proposition 36 that the solution g of (128) satisfies

$$\|g(T,\cdot)\|_{L^2(\mathbb{T})} \leqslant C\big(\|g_1\|_{L^2(q_T)} + \|\partial_x^{d_2} g_2\|_{H^{-2d_2+1}(q_T)}\big). \tag{129}$$

By using the fact that $\widehat{A_{22}}$ is a companion matrix, see (104), for every $i \in \{2, \dots, d_2\}$, the i-th equation of (128) is

$$\partial_t g_2^{i-1} - \partial_x^2 g_2^{i-1} + \partial_x g_2^i + b_{i-1}\partial_x g_1 + a_{i-1} g_1 = 0, \qquad (a_{i-1}, b_{i-1}) \in \mathbb{R}^2. \tag{130}$$

By applying ∂_x^{i-1} to (130) with $i \in \{2, \dots, d_2\}$, we get that there exists $C > 0$ such that

$$\begin{aligned} &\|\partial_x^i g_2^i\|_{H^{-2i+1}(q_T)} \\ &\qquad \leqslant C\big(\|(\partial_t - \partial_x^2)\partial_x^{i-1} g_2^{i-1}\|_{H^{-2i+1}(q_T)} + \|(b_{i-1}\partial_x^i + a_{i-1}\partial_x^{i-1}) g_1\|_{H^{-2i+1}(q_T)}\big), \end{aligned}$$

therefore we have

$$\|\partial_x^i g_2^i\|_{H^{-2i+1}(q_T)} \leqslant C\big(\|\partial_x^{i-1} g_2^{i-1}\|_{H^{-2(i-1)+1}(q_T)} + \|g_1\|_{L^2(q_T)}\big). \tag{131}$$

We deduce from (129) and (131) that

$$\begin{aligned} \|g(T,\cdot)\|_{L^2(\mathbb{T})} &\leqslant C\big(\|g_1\|_{L^2(q_T)} + \|\partial_x^{d_2} g_2\|_{H^{-2d_2+1}(q_T)}\big) \\ &\leqslant C\Big(\|g_1\|_{L^2(q_T)} + \sum_{i=1}^{d_2} \|\partial_x^i g_2^i\|_{H^{-2i+1}(q_T)}\Big) \\ &\leqslant C\big(\|g_1\|_{L^2(q_T)} + \|\partial_x g_2^1\|_{H^{-1}(q_T)}\big). \end{aligned} \tag{132}$$

By using the fact that $\widehat{A_{21}}$ is the first vector of the canonical basis of $\mathbb{R}^{d_2}$, see (126), the first equation of (128) is

$$\partial_t g_1 - A'\partial_x g_1 + K_{11} g_1 + \partial_x g_2^1 = 0.$$

We obtain

$$\|\partial_x g_2^1\|_{H^{-1}(q_T)} \leqslant C\|g_1\|_{L^2(q_T)}. \tag{133}$$

So, we deduce from (132) and (133) the observability inequality

$$\|g(T,\cdot)\|_{L^2(\mathbb{T})} \leqslant C\|g_1\|_{L^2(q_T)},$$

which concludes the proof of Theorem 3 in the case $d_1 = 1$. $\square$

7. Parabolic control

The goal of this section is to prove Theorem 4 and to illustrate the necessity of a regularity assumption on the initial condition.

7.1. A regularity assumption is necessary. — We consider for $\lambda \in \mathbb{R}^*$ the system

$$\begin{cases} \partial_t \widetilde{f}_1 + \lambda\partial_x \widetilde{f}_1 + \partial_x \widetilde{f}_2 = 0, & \text{in } (0,T)\times\mathbb{T}, \\ \partial_t \widetilde{f}_2 - \partial_x^2 \widetilde{f}_2 + \lambda\partial_x \widetilde{f}_2 = v(t,x), & \text{in } (0,T)\times\mathbb{T}, \end{cases} \tag{134}$$

i.e., $\omega = \mathbb{T}$, $d = 2$, $m = 1$,

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad A' = (\lambda), \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

that satisfies (H.3), (H.4) and the Kalman condition (12) because $A_{12} = 1$. By Theorem 4, any initial condition $f_0 = (f_{01}, f_{02}) \in H^2_{\mathrm{m}} \times H^2(\mathbb{T})$ is null-controllable. The following statement illustrates that

– a regularity assumption on f_{01} is necessary for the null-controllability,
– that given by Theorem 4 is sufficient but may not be necessary.

Proposition 40. — *An initial condition $f_0 = (f_{01}, f_{02}) \in L^2_{\mathrm{m}}(\mathbb{T}) \times L^2(\mathbb{T})$ is null-controllable with $v \in L^2((0,T) \times \mathbb{T})$ if and only if $f_{01} \in H^1(\mathbb{T})$.*

Remark 41. — Similar problems of regularity between initial data and control have already been noticed in the context of transport systems, see [1, Rem. 5].

Proof. — In the proof, we use the notation $Q_T = (0,T) \times \Omega$. The change of variable

$$\widetilde{f_j}(t,x) = f_j(t, x - \lambda t), \qquad v(t,x) = u(t, x - \lambda t)$$

leads to

$$\begin{cases} \partial_t f_1 - \partial_x f_2 = 0, & \text{in } (0,T) \times \mathbb{T}, \\ \partial_t f_2 - \partial_x^2 f_2 = u(t,x), & \text{in } (0,T) \times \mathbb{T}. \end{cases} \tag{135}$$

The null-controllability of $(\widetilde{f_1}, \widetilde{f_2})$ with control $v \in L^2(Q_T)$ is equivalent to the null-controllability of (f_1, f_2) with control $u \in L^2(Q_T)$. On Fourier components, Equation (135) gives the ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \widehat{f_1}(t,n) = \mathrm{i}n \widehat{f_2}(t,n), \\ \dfrac{\mathrm{d}}{\mathrm{d}t} \widehat{f_2}(t,n) = -n^2 \widehat{f_2}(t,n) + \widehat{u}(t,n). \end{cases} \tag{136}$$

Let $f_0 = (f_{01}, f_{02}) \in L^2_{\mathrm{m}}(\mathbb{T}) \times L^2(\mathbb{T})$. The solution writes

$$\begin{aligned} \widehat{f_2}(t,n) &= \widehat{f_{02}}(n) \mathrm{e}^{-n^2 t} + \int_0^t \mathrm{e}^{-n^2(t-\tau)} \widehat{u}(\tau,n)\,\mathrm{d}\tau, \\ \widehat{f_1}(t,n) &= \widehat{f_{10}}(n) + \mathrm{i}n \int_0^t \widehat{f_2}(\tau,n)\,\mathrm{d}\tau \\ &= \widehat{f_{01}}(n) + \mathrm{i}/n(1 - \mathrm{e}^{-n^2 T}) \widehat{f_{02}}(n) + \mathrm{i}n \int_0^t \widehat{u}(\tau,n) \frac{1 - \mathrm{e}^{-n^2(t-\tau)}}{n^2}\,\mathrm{d}\tau. \end{aligned}$$

thus the relation $f(T) = 0$ is equivalent to the moment problem

$$\begin{aligned} \int_0^T \mathrm{e}^{-n^2(T-\tau)} \widehat{u}(\tau,n)\,\mathrm{d}\tau = -\widehat{f_{02}}(n) \mathrm{e}^{-n^2 T}, \quad \forall\, n \in \mathbb{Z}, \\ \int_0^T \widehat{u}(\tau,n)\,\mathrm{d}\tau = \mathrm{i}n \widehat{f_{01}}(n) - \widehat{f_{02}}(n), \quad \forall\, n \in \mathbb{Z} \smallsetminus \{0\}. \end{aligned} \tag{137}$$

Note that the assumption $\int_{\mathbb{T}} f_{01} = 0$ implies $\int_{\mathbb{T}} f_1(t) = 0$ for every $t > 0$ thus the null-controllability of this component does not require any condition on the control u.

Necessary condition. — We assume $f_0 = (f_{01}, f_{02})$ null-controllable with a control $u \in L^2(Q_T)$ and we prove that $f_{01} \in H^1(\mathbb{T})$. By the Bessel-Parseval equality and Cauchy-Schwarz inequality,

$$\begin{aligned}\|u\|^2_{L^2(Q_T)} &= \sum_{n\in\mathbb{Z}} \int_0^T |\widehat{u}(t,n)|^2 \,\mathrm{d}t \geqslant \sum_{n\in\mathbb{Z}} \frac{1}{T} \left| \int_0^T \widehat{u}(t,n)\,\mathrm{d}t \right|^2 \\ &\geqslant \sum_{n\in\mathbb{Z}} \frac{1}{T} \left| \mathrm{i}n\widehat{f}_{01}(n) - \widehat{f}_{02}(n) \right|^2 = \frac{1}{T} \|\partial_x f_{01} - f_{02}\|^2_{L^2(\mathbb{T})}\end{aligned}$$

thus $f_{01} \in H^1(\mathbb{T})$.

Sufficient condition. — We assume $f_0 = (f_{01}, f_{02}) \in H^1_{\mathrm{m}} \times L^2(\mathbb{T})$ and we construct a control $u \in L^2((0,T)\times\mathbb{T})$ that steers this initial condition to 0.

Let G_n be the Gramian matrix, in $L^2(0,T)$, of the family $(w_{1,n}, w_{2,n})$ where $w_{1,n}\colon \tau \mapsto n\mathrm{e}^{-n^2(T-\tau)}$ and $w_{2,n} : \tau \mapsto 1$, i.e., $(G_n)_{i,j} = \int_0^T w_{i,n}(\tau) w_{j,n}(\tau)\,\mathrm{d}\tau$ for every $1 \leqslant i,j \leqslant 2$. Then G_n is invertible for every $n \in \mathbb{Z}\smallsetminus\{0\}$ (because it is the Gramian matrix of a linearly independent family) and, when $|n| \to \infty$,

$$G_n \sim \begin{pmatrix} 1/2 & 1/n \\ 1/n & T \end{pmatrix}$$

thus there exists $C > 0$ such that, for every $n \in \mathbb{Z}\smallsetminus\{0\}$, $\|G_n^{-1}\| \leqslant C$. We take

$$u(\tau,x) = -\frac{1}{T}\,\widehat{f}_{02}(0) + \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} (\alpha_n w_{1,n}(\tau) + \beta_n w_{2,n}(\tau))\mathrm{e}^{\mathrm{i}nx}$$

where

$$\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} := G_n^{-1} \begin{pmatrix} -n\widehat{f}_{02}(n)\mathrm{e}^{-n^2T} \\ \mathrm{i}n\widehat{f}_{01}(n) - \widehat{f}_{02}(n)\end{pmatrix}. \tag{138}$$

By Bessel-Parseval equality, we have for various positive constants C depending on T,

$$\begin{aligned}\|u\|^2_{L^2(Q_T)} &= \frac{1}{T}|\widehat{f}_{02}(0)|^2 + \int_0^T \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} |\alpha_n w_{1,n}(t) + \beta_n w_{2,n}(t)|^2 dt \\ &\leqslant \frac{1}{T}|\widehat{f}_{02}(0)|^2 + C \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \bigl(|\alpha_n|^2 + |\beta_n|^2\bigr) \\ &\leqslant \frac{1}{T}|\widehat{f}_{02}(0)|^2 + C \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \bigl(|n\widehat{f}_{02}(n)\mathrm{e}^{-n^2T}|^2 + |\mathrm{i}n\widehat{f}_{01}(n) - \widehat{f}_{02}(n)|^2\bigr) \\ &\leqslant C\bigl(\|f_{01}\|^2_{H^1(\mathbb{T})} + \|f_{02}\|^2_{L^2(\mathbb{T})}\bigr) < \infty.\end{aligned}$$

Thus $u \in L^2(Q_T)$. Note that the moment problem (137) can equivalently be written

$$(139)\qquad \begin{aligned} &\int_0^T \widehat{u}(\tau,0)\,\mathrm{d}\tau = -\widehat{f}_{02}(0)\,,\\ &\int_0^T w_{1,n}(\tau)\widehat{u}(\tau,n)\,\mathrm{d}\tau = -n\widehat{f}_{02}(n)\mathrm{e}^{-n^2T}, \quad \forall\, n\in\mathbb{Z}\smallsetminus\{0\},\\ &\int_0^T w_{2,n}(\tau)\widehat{u}(\tau,n)\,\mathrm{d}\tau = \mathrm{i}n\widehat{f}_{01}(n) - \widehat{f}_{02}(n), \quad \forall\, n\in\mathbb{Z}\smallsetminus\{0\}. \end{aligned}$$

Therefore, by (138), u solves (137). □

7.2. Proof of Theorem 4. — The Kalman rank condition (12) is a necessary condition for null-controllability of (10) by the same arguments as in Section 5.1. Thus we only explain how to complete the proof of Theorem 2 to prove that it is a sufficient condition for null-controllability of (10). We introduce the space

$$(140)\qquad \mathbf{F_1} := H^{d_1+1}_{\mathrm{m}}(\mathbb{T})^{d_1} \times H^{d_1+1}(\mathbb{T})^{d_2},$$

equipped with the scalar product of $H^{d_1+1}(\mathbb{T})^d$ and the space

$$(141)\qquad \widetilde{\mathbf{F_1}} := L^2_{\mathrm{m}}(\mathbb{T})^{d_1} \times L^2(\mathbb{T})^{d_2},$$

equipped with the scalar product of $L^2(\mathbb{T})^d$.

The null-controllability of the system (10) in $\mathbf{F_1}$ with control of the form $(0,u_2)\in\{0\}^{d_1}\times L^2(q_T)^{d_2}$ is equivalent to the following observability inequality: for every $T>T^*$, there exists $C>0$ such that, for every $g_0\in\widetilde{\mathbf{F_1}}$, the solution of the adjoint system (31) satisfies

$$(142)\qquad \|g(T,\cdot)\|^2_{H^{-(d_1+1)}(\mathbb{T})^d} \leqslant C\int_0^T\int_\omega |g_2(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

where $g_2(t,x)\in\mathbb{C}^{d_2}$ is made of the last d_2 components of $g(t,x)$.

Proof of the equivalence between the null-controllability in $\mathbf{F_1}$ *and the observability inequality* (142)

We apply the duality lemma 14 with

$$\Phi_2\colon f_0\in\mathbf{F_1}\longmapsto \mathrm{e}^{-T\mathscr{L}}f_0\in\widetilde{\mathbf{F_1}},$$
$$\Phi_3\colon u_2\in L^2(q_T)^{d_2}\longmapsto S(T;0,(0,u_2))\in\widetilde{\mathbf{F_1}}.$$

Note that the mean value of the d_1 first components is indeed zero. The null-controllability result in $\mathbf{F_1}$ is equivalent to the inclusion $\mathrm{Im}(\Phi_2)\subset\mathrm{Im}(\Phi_3)$, thus to the existence of a constant $C>0$ such that for every $g_T\in\widetilde{\mathbf{F_1}}$

$$(143)\qquad \|\Phi_2^*(g_T)\|_{H^{d_1+1}(\mathbb{T})^d} \leqslant C\|\Phi_3^*(g_T)\|_{L^2(q_T)^{d_2}}.$$

We compute the adjoint operators of Φ_2 and Φ_3 thanks to the duality relation between the solution f of (Sys) and the solution $\varphi(\cdot)=g(T-\cdot)$ of the adjoint system (31):

$$(144)\qquad \langle f(T),\varphi(T)\rangle_{L^2(\mathbb{T})^d} = \langle f(0),\varphi(0)\rangle_{L^2(\mathbb{T})^d} + \int_0^T\int_\omega \langle u_2(t,x),\varphi_2(t,x)\rangle\,\mathrm{d}t\,\mathrm{d}x.$$

First, $\Phi_3^*(g_T)$ is the restriction of the d_2-last components of $\mathrm{e}^{(t-T)\mathscr{L}^*}g_T$ to $[0,T]\times\omega$. Then, by (144) and Lemma 33 (working as in the proof of Corollary 32), the left-hand side of (143) is

$$\|\Phi_2^*(g_T)\|_{H^{d_1+1}(\mathbb{T})^d} = \|\mathrm{e}^{-T\mathscr{L}^*}g_T\|_{H^{-(d_1+1)}(\mathbb{T})^d}.$$

Thus the inequality (143) is indeed the observability inequality (142). □

By using the strategy developed in Section 6, we claim that, in the case $d_2=1$, it is sufficient to prove the following result in order to prove the observability inequality (142).

Proposition 42. — *For every $T>T^*$, there exists $C>0$ such that for every $g_0\in\widetilde{\mathbf{F_1}}$, the solution g of the adjoint system* (31) *satisfies*

$$\|g(T,\cdot)\|^2_{H^{-(d_1+1)}(\mathbb{T})^d} \leqslant C\big(\|\partial_x^{d_1}g_1\|^2_{H^{-(d_1+1)}(q_T)}+\|g_2\|^2_{L^2(q_T)}\big). \tag{145}$$

The observability inequality (145) has to be compared to the observability inequality (122) in Section 6. Roughly speaking, the term $\|\partial_x^{d_1}g_1\|_{H^{-(d_1+1)}(q_T)}$ comes from the fact that we will perform (d_1-1) steps of elimination, each of them "costs" one derivative (instead of two in Section 6.2) because we will use transport equations which are of order one in time and space (instead of parabolic equations which are of order two in space variable). The last step of elimination "costs" two derivatives because we will use a heat equation which is of order one in time and two in space. This explains the number $(d_1-1)+2=d_1+1$ derivatives. By adapting the arguments of Section 5.2.2, we can also treat the case $d_2>1$.

In order to prove Proposition 42, by duality (a simple adaptation of the proof that Proposition 36 and Proposition 37 are equivalent), it is sufficient to establish the following null-controllability result.

Proposition 43. — *For every $f_0\in\mathbf{F_1}$, there exists*

$$u=(u_{\mathrm{h}},u_{\mathrm{p}})\in(H_0^{2d_1+1}(q_T))^{d_1}\times L^2(q_T)^{d_2}$$

such that $S(T,f_0,(\partial_x^{d_1}u_{\mathrm{h}},u_{\mathrm{p}}))=0$.

The proof of this result is an adaptation of the proof of Theorem 2:

– we prove that parabolic high frequencies are null-controllable,

– we prove that hyperbolic high frequencies are null-controllable,

– we combine these two propositions to prove that high frequencies are null-controllable,

– we finally deal with low frequencies.

For the first point, we just need a special case of the corresponding result that was used in the proof of Theorem 2, i.e., Proposition 22.

Proposition 44. — *If n_0 is large enough, there exists a continuous operator*

$$\begin{aligned}\mathscr{U}^{\mathrm{p},\sharp}\colon \mathbf{F_1}\times H_0^{2d_1+1}((0,T')\times\omega)^{d_1} &\longrightarrow C_c^\infty((T',T)\times\omega)^{d_2}\\ (f_0,u_{\mathrm{h}}) &\longmapsto u_{\mathrm{p}},\end{aligned}$$

that associates with any $(f_0, u_{\mathrm{h}}) \in \mathbf{F_1} \times H_0^{2d_1+1}((0,T') \times \omega)^{d_1}$ *a control* $u_{\mathrm{p}} = \mathscr{U}^{\mathrm{p},\sharp}(f_0, u_{\mathrm{h}})$ *such that*

$$\Pi^{\mathrm{p}} S(T; f_0, (\partial_x^{d_1} u_{\mathrm{h}}, u_{\mathrm{p}})) = 0.$$

Proof. — Proposition 44 is a consequence of Proposition 22 because

$$\mathbf{F_1} \times H_0^{2d_1+1}((T',T) \times \omega)^{d_2}$$

is continuously embedded in $L^2(\mathbb{T})^d \times L^2((T',T)\times\omega)^{d_2}$ and $\partial_x^{d_1} u_{\mathrm{h}} \in L^2((0,T')\times\omega)^{d_1}$ for every $u_{\mathrm{h}} \in H_0^{2d_1+1}((0,T') \times \omega)^{d_1}$. □

For the second point, we will prove the following adaptation of Proposition 21.

Proposition 45. — *If* n_0 *is large enough, there exists a continuous operator*

$$\begin{aligned} \mathscr{U}^{\mathrm{h},\sharp} \colon \mathbf{F_1} \times H_0^{2d_1+1}((T',T) \times \omega)^{d_2} &\longrightarrow H_0^{2d_1+1}((0,T') \times \omega)^{d_1} \\ (f_0, u_{\mathrm{p}}) &\longmapsto u_{\mathrm{h}}, \end{aligned}$$

that associates with any $(f_0, u_{\mathrm{p}}) \in \mathbf{F_1} \times H_0^{2d_1+1}((T',T) \times \omega)^{d_2}$ *a control* $u_{\mathrm{h}} = \mathscr{U}^{\mathrm{h},\sharp}(f_0, u_{\mathrm{p}})$ *such that*

$$\Pi^{\mathrm{h}} S(T; f_0, (\partial_x^{d_1} u_{\mathrm{h}}, u_{\mathrm{p}})) = 0. \tag{146}$$

While the ideas of the proof are the same as for Proposition 21, the proof of this Proposition is technically more delicate, as we have to build regular controls, and, on the observability side, deal with the (slightly impractical) H_0^s and H^{-s} norms. We postpone the proof to the next subsection. For now, let us assume Proposition 45 holds true, and finish the proof of Theorem 4.

We now combine Propositions 44 and 45 with the Fredholm alternative, as in the proof of Proposition 20, to prove that high frequencies are null-controllable. That is to say, we get the following adaptation of Proposition 20.

Proposition 46. — *There exist a closed subspace* $\mathscr{G}^\sharp$ *of* $\mathbf{F}_1$ *with finite codimension and a continuous operator*

$$\begin{aligned} \mathscr{U}^{\sharp} \colon \mathscr{G}^\sharp &\longrightarrow H_0^{2d_1+1}(q_T)^{d_1} \times H_0^{2d_1+1}(q_T)^{d_2} \\ f_0 &\longmapsto (u_{\mathrm{h}}, u_{\mathrm{p}}), \end{aligned}$$

that associates with each $f_0 \in \mathscr{G}^\sharp$ *a pair of controls* $\mathscr{U}^\sharp f_0 = (u_{\mathrm{h}}, u_{\mathrm{p}})$ *such that*

$$\forall\, f_0 \in \mathscr{G}^\sharp,\ \Pi S(T; f_0, (\partial_x^{d_1} u_{\mathrm{h}}, u_{\mathrm{p}})) = 0. \tag{147}$$

The last step consists in showing that the null-controllability result of Proposition 43 is true at the low frequency-level, as we have already shown for Theorem 2 in Section 4.5. All the steps of Section 4.5 remain unchanged except Step 6. Indeed, the unique continuation argument transforms into: if $f(t,\cdot) = \mathrm{e}^{tM} f_0$ with $(\partial_x^{d_1} f_1, f_2) = (0,0)$ in $(0,\varepsilon) \times \omega$ then $(\partial_x^{d_1} f_{01}, f_{02}) = (0,0)$ thanks to the spectral inequality of Lebeau-Robbiano (90), that is to say, $f_0 = 0$ because $\int_{\mathbb{T}} f_{01}(x)\,\mathrm{d}x = 0$.

This concludes the proof of Proposition 43 thus the proof of Proposition 42. □

7.3. Proof of Proposition 45. — The proof of Proposition 45 is an adaptation of that of Proposition 21, with the following changes:

– we deal with the fact that we want a control of the form $(\partial_x^{d_1} u_{\mathrm{h}}, 0)$,

– we adapt the duality argument to take into account the regularity of the controls that we want (it involves some H^{-s} norms),

– we adapt all the inequalities to replace the relevant L^2 norms by H^{-s} norms,

– to build regular controls of the simple transport equation $\partial_t f + \mu \partial_x f = 0$, we use [1].

Step 1: reduction to an exact controllability problem. — We claim that in order to prove Proposition 45, we only have to prove the following exact controllability result.

Proposition 47. — *If n_0 is large enough, then for every $T' > T^*$, there exists a continuous operator*

$$\begin{aligned} \underline{\mathscr{U}}_{T'}^{\mathrm{h};\sharp} \colon F^{\mathrm{h}} \cap H^{2d_1+1}(\mathbb{T})^d &\longrightarrow H_0^{2d_1+1}(q_{T'})^{d_1} \\ f_{T'} &\longmapsto u_{\mathrm{h}}, \end{aligned}$$

that associates with any $f_{T'} \in F^{\mathrm{h}} \cap H^{2d_1+1}(\mathbb{T})^d$, a control $\underline{\mathscr{U}}_{T'}^{\mathrm{h};\sharp}(f_{T'}) = u_{\mathrm{h}}$ such that

$$\Pi^{\mathrm{h}} S(T'; 0, (u_{\mathrm{h}}, 0)) = f_{T'}.$$

Indeed, by the choice of support in time of the controls, and by the reversibility of $\mathrm{e}^{-t\mathscr{L}^{\mathrm{h}}}$ (see Section 4.3.1 for the details), the relation (146) is equivalent to

$$\Pi^{\mathrm{h}}(S(T'; 0, (\partial_x^{d_1} u_{\mathrm{h}}, 0))) = -\mathrm{e}^{(T-T')\mathscr{L}^{\mathrm{h}}} \Pi^{\mathrm{h}} S(T; f_0, (0, u_{\mathrm{p}})).$$

Note that functions in F^{h} have zero mean (see the definition (42) of F^{h}). Thus, $\partial_x^{d_1}$ is invertible on F^{h}, and its inverse $\partial_x^{-d_1}$ is, on the Fourier side, the multiplication by $(\mathrm{i}n)^{-d_1}$. Moreover, the operator ∂_x commute with Π^{h} and the semi-group $\mathrm{e}^{-t\mathscr{L}}$. So the relation (146) is equivalent to

$$\Pi^{\mathrm{h}}(S(T'; 0, (u_{\mathrm{h}}, 0))) = -\partial_x^{-d_1} \mathrm{e}^{(T-T')\mathscr{L}^{\mathrm{h}}} \Pi^{\mathrm{h}} S(T; f_0, (0, u_{\mathrm{p}})) =: K(f_0, u_{\mathrm{p}}). \tag{148}$$

So, if Proposition 47 holds, we may choose (assuming it makes sense)

$$u_{\mathrm{h}} := \underline{\mathscr{U}}_{T'}^{\mathrm{h},\sharp}(K(f_0, u_{\mathrm{p}})).$$

Thus, to end this first step, we just have to check that the right-hand side $K(f_0, u_{\mathrm{p}})$ of (148) is indeed in $F^{\mathrm{h}} \cap H^{2d_1+1}(\mathbb{T})^d$.

The projection Π^{h} has range F^{h}, and $\mathrm{e}^{t\mathscr{L}^{\mathrm{h}}}$ sends F^{h} to itself, as do $\partial_x^{-d_1}$. So $K(f_0, u_{\mathrm{p}})$ belongs to F^{h}.

The group $\mathrm{e}^{t\mathscr{L}^{\mathrm{h}}}$ sends every $H^s(\mathbb{T})^d$ into itself (see Remark 13). Since Π^{h} is just the multiplication on the Fourier side by $P^{\mathrm{h}}(\mathrm{i}/n)$, the operator Π^{h} also sends every $H^s(\mathbb{T})^d$ into itself. Thus, we just have to check that $S(T, f_0, (0, u_{\mathrm{p}})) = \mathrm{e}^{-T\mathscr{L}} f_0 + S(T, 0, (0, u_{\mathrm{p}})) \in H^{d_1+1}(\mathbb{T})^d$ because $\partial_x^{-d_1}$ sends $H^{d_1+1}(\mathbb{T})^d$ into $H^{2d_1+1}(\mathbb{T})^d$.

– The function f_0 belongs to $H^{d_1+1}(\mathbb{T})$ by hypothesis, so $\mathrm{e}^{-T\mathscr{L}} f_0$ also belongs to $H^{d_1+1}(\mathbb{T})$ (see Remark 13).

– The parabolic control u_p belongs to $H_0^{2d_1+1}((T',T)\times\omega)^{d_2}$ by hypothesis, thus for almost every $t\in(0,T)$, $(0,u_\mathrm{p})(t,\cdot)$ belongs to $H^{2d_1+1}(\mathbb{T})$ and thus

$$S(T;0,(0,u_\mathrm{p})) = \int_0^T \mathrm{e}^{-(T-t)\mathscr{L}}(0,u_\mathrm{p})(t)\,\mathrm{d}t \in H^{2d_1+1}(\mathbb{T})^d.$$

This concludes this first step.

Step 2: Observability inequality associated to the controllability problem of Proposition 47. — Let

$$\Phi_2 := \Pi^\mathrm{h}\circ\iota_{2d_1+1}\colon H^{2d_1+1}(\mathbb{T})^d \longrightarrow L^2(\mathbb{T})^d$$

be the restriction of Π^h to $H^{2d_1+1}(\mathbb{T})^d$ and

$$\Phi_3 := \Pi^\mathrm{h}\circ\mathscr{F}_{T'}\circ(\iota_{2d_1+1},0)\colon H_0^{2d_1+1}(q_{T'})^{d_1} \longrightarrow L^2(\mathbb{T})^d,$$

where $(\iota_{2d_1+1},0)$ stands for the map $u_\mathrm{h}\in H_0^{2d_1+1}(q_{T'})^{d_1}\mapsto(u_\mathrm{h},0)\in L^2(q_{T'})^d$. Note that Φ_2 and Φ_3 are continuous.

The controllability problem of Proposition 47 is equivalent to the inclusion $\mathrm{Im}(\Phi_2)\subset\mathrm{Im}(\Phi_3)$. Therefore, according to the duality lemma 14, it is equivalent to the following inequality: there exists $C>0$ such that for every $g_0\in L^2(\mathbb{T})^d$, $\|\Phi_2^* g_0\|_{H^{2d_1+1}(\mathbb{T})^d}\leqslant C\|\Phi_3^* g_0\|_{H_0^{2d_1+1}(q_{T'})^d}$. Since Π^{h^*} is a projection on $\widetilde{F^\mathrm{h}}$, since $\mathscr{F}_{T'}^* g_0$ is the restriction of the first d_1 components of $\mathrm{e}^{-(T'-t)\mathscr{L}^*}g_0$ to $q_{T'}$, and since ι_s^* is an isometry between H_0^s and H^{-s},[10] this inequality reads: there exists $C>0$ such that for every $g_0\in\widetilde{F}^\mathrm{h}$, the solution $g=\mathrm{e}^{-t\mathscr{L}^*}g_0$ of the adjoint system (31) satisfies

$$\|g_0\|_{H^{-2d_1-1}(\mathbb{T})^d}\leqslant C\|g_1\|_{H^{-2d_1-1}(q_{T'})^{d_1}}, \tag{149}$$

where g_1 are the first d_1 components of g.

Let $g_0\in\widetilde{F}^\mathrm{h}$. For the remaining of this proof, we use the notation of Section 4.3.2, and in particular we introduce the decompositions (61) and (62). In the following arguments, the constants C do not depend on g_0.

Step 3. — We prove the observability inequality (149) assuming that, for every $\mu\in\mathrm{Sp}(A')$, there exists $C>0$ such that the solution $G_\mu^\flat$ of (70) satisfies

$$\|G_\mu(0,\cdot)\|_{H^{-(2d_1+1)}(\mathbb{T})} = \|G_\mu^\flat(0,\cdot)\|_{H^{-(2d_1+1)}(\mathbb{T})}\leqslant C\|G_\mu^\flat\|_{H^{-(2d_1+1)}(q_T)}. \tag{150}$$

We will prove (150) in Step 4.

We proceed as in the proof given in Section 4.3.2. By the explicit expression (66) of S_μ and Bessel-Parseval identity, there exists $C=C(T')$ independent of g_0 such that

$$\|S_\mu\|_{L^\infty((0,T'),H^{-(2d_1+1)}(\mathbb{T})^d)}\leqslant C\|g(0,\cdot)\|_{H^{-(2d_1+2)}(\mathbb{T})^d}. \tag{151}$$

[10]See Lemma 33, and also recall that because $\mathbb{T}$ has no boundary $H_0^s(\mathbb{T})=H^s(\mathbb{T})$.

Using the Duhamel formula, we obtain that the function $\widetilde{G}_\mu$ defined by (68) satisfies

$$\begin{aligned}(152)\quad &\|\widetilde{G}_\mu - G^\flat_\mu\|_{L^\infty((0,T'),H^{-(2d_1+1)}(\mathbb{T})^d)}\\ &\qquad\leqslant C\|\mathrm{e}^{tR^{\mathrm{h}}_\mu(0)^*}S_\mu\|_{L^1((0,T'),H^{-(2d_1+1)}(\mathbb{T})^d)} \leqslant C\|g_0\|_{H^{-(2d_1+2)}(\mathbb{T})^d}.\end{aligned}$$

By (150), the triangular inequality, (68) and (152), we deduce that

$$\begin{aligned}(153)\quad \|G_\mu(0,\cdot)\|_{H^{-(2d_1+1)}(\mathbb{T})^d} &\leqslant C\big(\|\widetilde{G}_\mu\|_{H^{-(2d_1+1)}(q_{T'})^d} + \|\widetilde{G}_\mu - G^\flat_\mu\|_{H^{-(2d_1+1)}(q_{T'})^d}\big)\\ &\leqslant C\big(\|G_\mu\|_{H^{-(2d_1+1)}(q_{T'})^d} + \|g_0\|_{H^{-(2d_1+2)}(\mathbb{T})^d}\big).\end{aligned}$$

Using Bessel-Parseval identity and the decomposition (73), we obtain

$$(154)\qquad \|G_\mu - P^{\mathrm{h}}_\mu(0)^*g\|_{L^\infty((0,T'),H^{-(2d_1+1)}(\mathbb{T})^d)} \leqslant C\|g_0\|_{H^{-(2d_1+2)}(\mathbb{T})^d}.$$

We deduce from (153), the triangular inequality and (154)) that

$$\|G_\mu(0,\cdot)\|_{H^{-(2d_1+1)}(\mathbb{T})^d} \leqslant C\big(\|P^{\mathrm{h}}_\mu(0)^*g\|_{H^{-(2d_1+1)}(q_{T'})^d} + \|g_0\|_{H^{-(2d_1+2)}(\mathbb{T})^d}\big).$$

Taking into account that $P^{\mathrm{h}}_\mu(0)^* = P^{\mathrm{h}}_\mu(0)^*P^{\mathrm{h}}(0)^*$, we get[(11)]

$$\|P^{\mathrm{h}}_\mu(0)^*g\|_{H^{-(2d_1+1)}(q_{T'})^d} \leqslant |P^{\mathrm{h}}_\mu(0)^*|\|P^{\mathrm{h}}(0)^*g\|_{H^{-(2d_1+1)}(q_{T'})^d} \leqslant C\|g_1\|_{H^{-(2d_1+1)}(q_{T'})^{d_1}}.$$

Using (62), the triangular inequality and the previous two estimates, we obtain

$$\begin{aligned}(155)\quad \|g_0\|_{H^{-(2d_1+1)}(\mathbb{T})^d} &\leqslant \sum_{\mu\in\mathrm{Sp}(A')}\|G_\mu(0,\cdot)\|_{H^{-(2d_1+1)}(\mathbb{T})^d}\\ &\leqslant C\big(\|g_1\|_{H^{-(2d_1+1)}(q_{T'})^{d_1}} + \|g_0\|_{H^{-(2d_1+2)}(\mathbb{T})^d}\big).\end{aligned}$$

Proceeding as in the end of the proof given in Section 4.3.2, the inequality (155), together with a compactness-uniqueness argument, end Step 2.

Step 4: We prove that the solution $G^\flat_\mu$ of (70) *satisfies* (150). — By duality, it is actually enough to prove the following exact-controllability result.

PROPOSITION 48. — *Let $\omega = (a,b)$ and $T' > (2\pi - (b-a))/|\mu|$. For every $(f_0, f_{T'}) \in (H^{2d+1}(\mathbb{T})^d)^2$, there exists $u \in H^{2d_1+1}_0(q_T)^d$ such that the solution f of*

$$(156)\qquad \begin{cases}\partial_t f + \mu\partial_x f = u1_\omega & \text{in } Q_{T'},\\ f(0,\cdot) = f_0 & \text{in } \mathbb{T},\end{cases}$$

satisfies $f(T',\cdot) = f_{T'}$.

To prove Proposition 48, we will use the following lemma, which is an easy adaptation of [1, Lem. 2.6].

(11) Remark that if K is a matrix and $f \in (H^{-s})^d$, then $\|Kf\|_{H^{-s}} \leqslant |K|\|f\|_{H^{-s}}$. Indeed, denoting by $\langle\cdot,\cdot\rangle$ the duality between H^s_0 and H^{-s}, we have for every $g \in H^s_0$, $\langle Kf, g\rangle = \langle f, K^*g\rangle \leqslant \|f\|_{H^{-s}}\|K^*g_0\|_{H^s_0} \leqslant \|f\|_{H^{-s}}|K^*|\|g_0\|_{H^s_0}$, and taking the supremum over $\|g\|_{H^s_0} = 1$, we do have $\|Kf\|_{H^{-s}} \leqslant |K^*|\|f\|_{H^{-s}}$.

Lemma 49. — *Let $\omega = (a,b)$ and $T' > (2\pi - (b-a))/|\mu|$. Then, there exist $\delta > 0$ small enough and a cut-off function $\eta \in C^\infty([0,T'] \times [0,2\pi])$ with*

$$\eta = 0 \text{ in } [0,T'] \times [0,2\pi] \smallsetminus ((\delta, T'-\delta) \times (a+\delta, b-\delta)), \tag{157}$$

such that, for every $x \in [0,2\pi]$,

$$Q_x := \int_0^{T'} \eta(s, x+\mu s) ds \neq 0. \tag{158}$$

Remark 50. — We assumed that the function η is extended by 2π-periodicity in the spatial variable.

Now, we give the proof of Proposition 48 thanks to Lemma 49.

Proof of Proposition 48. — We take the control

$$u(t,x) = \eta(t,x) Q_{x-\mu t}^{-1}(f_{T'}(x+\mu(T'-t)) - f_0(x-\mu t)). \tag{159}$$

We easily check that the control u belongs to $H_0^k(q_T)$ by using the support of η (157), and the regularity of the three functions η, $f_{T'}$ and f_0. Let f be the solution of (156) with initial data f_0 and control u defined in (159). We just have to check that f satisfies $f(T', \cdot) = f_{T'}$. We write the solution along the characteristic, that is to say

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t, x+\mu t) = u(t, x+t) = \eta(t, x+\mu t) Q_x^{-1}(f_{T'}(x+\mu T') - f_0(x)).$$

By integrating in time between 0 and T' and by using the definition of Q_x (158), we obtain

$$f(T', \cdot + \mu T') - f(0, \cdot) = f_{T'}(\cdot + \mu T') - f_0(\cdot),$$

then $f(T', \cdot) = f_{T'}$ which concludes the proof of Proposition 48. □

This ends the proof of Proposition 47. □

Appendix. Pure transport solutions are not enough to disprove the observability inequality

Proposition 51. — *Let us assume that the $d \times d^2$ matrix*

$$(B|AB|\dots|A^{d-1}B)$$

has rank $= d$, or, equivalently, that there is no eigenvector of A^* in the kernel of B^* (see for instance [7, Lem. 1]*). Let $\mu \in \mathbb{R}$ and $T > 0$. There exists $C = C(\mu, T) > 0$*[12] *such that every solution of the adjoint system* (31) *of the form $g(t,x) = g_0(x - \mu t)$ satisfies $\|g(T,\cdot)\|_{L^2(\mathbb{T})^d} \leqslant C\|g\|_{L^2([0,T]\times\omega)^d}$.*

This statement shows that, for a dense set of matrices (A, B), pure transport solutions of the adjoint system (31) cannot be used to disprove the observability inequality (32), and thus the null-controllability of (Sys).

[12] With the help of Proposition 7, we could even prove that $C(\mu, T)$ can be chosen independently of μ.

Proof. — Let us note Sol_μ the set of solutions of the adjoint system (31) of the form $g_0(x-\mu t)$. Remark that according to the expression (33) of the solutions of the adjoint system, the relation $g_0 \in \mathrm{Sol}_\mu$ is equivalent to

$$\forall\, n \neq 0,\ nE(\mathrm{i}/n)^* \widehat{g}_0(n) = \mathrm{i}\mu \widehat{g}_0(n). \tag{160}$$

We claim that Sol_μ is finite dimensional. Indeed, if it is infinite dimensional, then, according to the relation (160), there is infinitely many n such that $\mathrm{i}\mu$ is an eigenvalue of $nE(\mathrm{i}/n)$. Let $(X_{n_k})_{k\geqslant 0}$ be an associated sequence of eigenvectors, chosen such that $|X_{n_k}| = 1$. Since the unit sphere of $\mathbb{C}^d$ is compact, we may assume that (X_{n_k}) converges to some X with $|X| = 1$. Then we have

$$n_k B^* X_{n_k} - \mathrm{i}A^* X_{n_k} + \frac{1}{n_k} K^* X_{n_k} = n_k E(\mathrm{i}/n_k)^* X_{n_k} = \mathrm{i}\mu X_{n_k} \xrightarrow[k\to+\infty]{} \mathrm{i}\mu X.$$

And since $-\mathrm{i}A^* X_{n_k} + (n_k)^{-1} K^* X_{n_k} \xrightarrow[k\to+\infty]{} -\mathrm{i}A^* X$, we must have $B^*X = 0$ and $A^*X = -\mu X$. But this is in contradiction with the hypothesis of the proposition. Therefore, Sol_μ is finite dimensional.

So, according to the description (160) of Sol_μ, there exists $N > 0$ such that every solution of the adjoint system (31) of the form $g_0(x-\mu t)$ has no frequencies higher than N: $\mathrm{Sol}_\mu \subset \mathrm{Span}\{e_n,\ |n| < N\}$. But finite linear combination of exponentials have the unique continuation property.(13) So the expressions $\|g_0(\cdot-\mu T)\|_{L^2(\mathbb{T})^d}$ and $\|g_0(x-\mu t)\|_{L^2([0,T]\times\omega)^d}$ both define a norm on Sol_μ. Since Sol_μ is finite dimensional, these two norms are equivalent. This proves the claimed inequality. ☐

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