NULL-CONTROLLABILITY FOR THE WEAKLY DISSIPATIVE FRACTIONAL HEAT EQUATIONS

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ABSTRACT. We prove that the fractional heat equations posed on the whole Euclidean space $\mathbb{R}^n$ and associated with the operators $(-\Delta)^{s/2}$ are exactly null-controllable from control supports which are sufficiently "exponentially thick", when $0 < s \leq 1$. This is a first positive null-controllability result in this weak dissipation regime. When $0 < s < 1$, we also give a necessary condition of the same form. This notion of exponential thickness is stronger than the geometric notion of thickness known to be a necessary and sufficient condition to obtain positive null-controllability results in the strong dissipation regime $s > 1$. Inspired by the construction of the Smith-Volterra-Cantor sets, we also provide examples of non-trivial exponentially thick control supports.

1. INTRODUCTION

This paper is part of the study of the null-controllability of the fractional heat equations

$$(E_s) \qquad \begin{cases} (\partial_t + (-\Delta)^{s/2})f(t,x) = \mathbb{1}_\omega u(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ f(0,\cdot) = f_0 \in L^2(\mathbb{R}^n), \end{cases}$$

where $s > 0$ is a positive real number and $\omega \subset \mathbb{R}^n$ is a measurable set with positive measure whose geometry is to be understood.

Although the null-controllability properties of parabolic equations posed on bounded domains of $\mathbb{R}^n$ are known for years [11], the same study for parabolic equations considered on the whole Euclidean space $\mathbb{R}^n$, as the equations (E_s), is quite recent. It follows from the works [1, 2, 3, 5, 6, 7, 13, 16, 17] that the null-controllability properties of such models, and also their approximate null-controllability or the stabilization properties, are associated with the geometric notion of thickness, defined as follows

Definition 1. Given $\gamma \in (0,1)$ and $L > 0$, the set $\omega \subset \mathbb{R}^n$ is said to be *γ-thick* at scale L when it is measurable and satisfies

$$\forall x \in \mathbb{R}^n, \quad \operatorname{Leb}(\omega \cap B(x,L)) \geq \gamma \operatorname{Leb}(B(x,L)),$$

where Leb denotes the Lebesgue measure in $\mathbb{R}^n$.

It was in particular proven in the works [1, 6, 17] that in the strong dissipation regime $s > 1$, the thickness is a necessary and sufficient geometric condition that allows to obtain positive null-controllability results for the equation (E_s). The present paper addresses the question of the null-controllability of the equation (E_s) in the low-dissipation regime $0 < s \leq 1$, for which very few results have been obtained so far. It is for example stated in the works [9, 12] that in dimension $n = 1$, the equation (E_s) is not null-controllable from a control support ω which is a strict open subset of $\mathbb{R}$. We refine this result by proving that the null-controllability properties of the equation (E_s) is associated with the following geometric condition for the control support ω, which we will call "exponential thickness"

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Definition 2. Given $\alpha > 0$, the set $\omega \subset \mathbb{R}^n$ is said to be *α-exponentially thick* when it is measurable and there exist $C > 0$, $c > 0$ and $L_0 > 0$ such that for every $0 < L < L_0$ and $x \in \mathbb{R}^n$,

$$\mathrm{Leb}(\omega \cap B(x, L)) \geq c\mathrm{e}^{-CL^{-\alpha}}\, \mathrm{Leb}(B(x, L)).$$

This definition can be rephrased as "ω is $c\mathrm{e}^{-CL^{-\alpha}}$-thick at every scale $0 < L < L_0$". The notion of exponential thickness is therefore far stronger than the notion of thickness. Our first result states that in the weak dissipation regime $0 < s < 1$, the fractional heat equation (E_s) is not null-controllable from a control support $\omega \subset \mathbb{R}^n$ which is not $2s/(1-s)$-exponentially thick. We also prove that in the case $0 < s \leq 1$, the equation (E_s) is exactly null-controllable from any α-exponentially thick set $\omega \subset \mathbb{R}^n$, with $0 < \alpha < s$, and in any positive time $T > 0$.

An example of α-exponential thick is of course the whole space $\mathbb{R}^n$, but it might be difficult to vizualize non trivial examples of sets satisfying this property. In dimension $n = 1$, inspired by the construction of the Cantor-Smith-Volterra sets, we give examples of subsets of $\mathbb{R}$ which are exactly α-exponentially thick, in the sense that they are not α'-exponentially thick with $\alpha' > \alpha$.

Outline of the work. In Section 2, we present in details the main results contained in this work. Section 3 is then devoted to prove that the fractional heat equations are null-controllable from exponentially thick control supports, the necessity of this geometric condition being investigated in Section 4. In Section 5, we construct examples of exponentially thick sets.

2. Main results

This section is devoted to present in details the main results contained in this work.

2.1. Null-controllability.

Let us first recall the definition of null-controllability.

Definition 3. Let $T > 0$ and $\omega \subset \mathbb{R}^n$ be a measurable set with positive measure. The equation (E_s) is said to be *null-controllable* from the control support ω in time $T > 0$ when for all $f_0 \in L^2(\mathbb{R}^n)$, there exists a control $u \in L^2((0, T) \times \omega)$ such that the mild solution of (E_s) satisfies $f(T, \cdot) = 0$.

By the Hilbert Uniqueness Method (see, e.g., [4, Theorem 2.44]), the null-controllability of the fractional heat equation (E_s) is equivalent to the observability of the fractional heat semigroup, whose we recall in the following definition.

Definition 4. Let $T > 0$, and let $\omega \subset \mathbb{R}^n$ be measurable. The fractional heat semigroup $(\mathrm{e}^{-t(-\Delta)^{s/2}})_{t\geq 0}$ is said to be exactly observable from the set ω at time T if there exists a positive constant $C_{\omega,T} > 0$ such that for all $g \in L^2(\mathbb{R}^n)$,

$$\|\mathrm{e}^{-T(-\Delta)^{s/2}} g\|^2_{L^2(\mathbb{R}^n)} \leq C_{\omega,T} \int_0^T \|\mathrm{e}^{-t(-\Delta)^{s/2}} g\|^2_{L^2(\omega)}\, \mathrm{d}t.$$

We first prove that the exponential thickness is a necessary geometric condition to prove exact null-controllability results for the equation (E_s) in the weak dissipation regime $0 < s < 1$.

Theorem 5. *Let $s \in (0, 1)$, $T > 0$ and $\omega \subset \mathbb{R}^n$ be measurable. If the fractional heat equation (E_s) is null-controllable from ω at time T, then ω is $2s/(1-s)$-exponentially thick.*

Remark 6. We do not consider the critical case $s = 1$, whose understanding remains an open problem.

We then prove that the exponential thickness is not only a necessary condition for the null-controllability of the equation (E_s), but is also a sufficient condition.

Theorem 7. *For all $s \in (0,1]$, the fractional heat equation (E_s) is null-controllable from any α-exponentially thick set $\omega \subset \mathbb{R}^n$, with $0 < \alpha < s$, and in any positive time $T > 0$. Moreover, there exists a positive constant $C > 0$ such that for all $T > 0$ and $f \in L^2(\mathbb{R}^n)$,*

$$\|\mathrm{e}^{-T(-\Delta)^{s/2}} f\|^2_{L^2(\mathbb{R}^n)} \leq C \exp\left(\frac{C}{T^{\alpha/(s-\alpha)}}\right) \frac{1}{T} \int_0^T \|\mathrm{e}^{-t(-\Delta)^{s/2}} f\|^2_{L^2(\omega)} \, \mathrm{d}t.$$

Remark 8. This provides a positive null-controllability result for the fractional heat equation (E_s) in the regime $0 < s \leq 1$ (notice that the critical case $s = 1$ is also allowed). Recall from [15, Theorem 3.8] or [1, Remark 1.13] that in the regime $s > 1$, the following exact observability estimates have been obtained for all thick set $\omega \subset \mathbb{R}^n$, $T > 0$ and $f \in L^2(\mathbb{R}^n)$

$$\|\mathrm{e}^{-T(-\Delta)^{s/2}} f\|^2_{L^2(\mathbb{R}^n)} \leq C \exp\left(\frac{C}{T^{1/(s-1)}}\right) \frac{1}{T} \int_0^T \|\mathrm{e}^{-t(-\Delta)^{s/2}} f\|^2_{L^2(\omega)} \, \mathrm{d}t.$$

The above observability estimate holds in particular for any α-exponential thick set $\omega \subset \mathbb{R}^n$.

2.2. **Examples.** To end this section, we present examples of exponentially thick control supports. Their construction is based on the one of the Smith-Volterra-Cantor sets, defined as follows

Definition 9 (Smith-Volterra-Cantor sets)**.** Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $0 < r_n < 1$. For $n \in \mathbb{N}$, let K_n be the closed subset of $[0,1]$, finite union of closed disjoint intervals, defined inductively by the following procedure.

- $K_0 := [0,1]$.
- If $K_n = \bigcup_k I_{nk}$, where the $(I_{nk})_k$ are disjoint closed intervals, remove from I_{nk} the middle part of size $r_n \operatorname{Leb}(I_{nk})$ and call the resulting sets[1] I'_{nk}. Then set $K_{n+1} := \bigcup_k I'_{nk}$.

Let $K := \bigcap_{n\in\mathbb{N}} K_n$. The set K is the Smith-Volterra-Cantor set associated to the sequence $(r_n)_n$.

Theorem 10. *Let $\alpha > 0$, $c \in (0,1)$ and $C > 0$. Set $r_n := c\exp(-C2^{n\alpha})$. Let K be the associated Smith-Volterra-Cantor set and $\omega := \mathbb{R} \setminus K$. There exist $c', C', L_0 > 0$ such that for every $0 < L < L_0$,*

$$c' \exp(-C'L^{-\alpha}) \leq \inf_{x\in\mathbb{R}} \frac{\operatorname{Leb}(\omega \cap B(x,L))}{\operatorname{Leb}(B(x,L))} \leq C' \exp(-c'L^{-\alpha}).$$

In other words, ω is α-exponentially thick, and not better.

The set ω defined in the statement of theorem 10 is α-exponentially thick, but does not have full measure. It is in fact the *simplest* of such sets that we could think of.

The ω defined in theorem 10 is a subset of $\mathbb{R}$. To get non-trivial α-exponentially thick sets in higher dimension, we can just take ω^n.

3. Sufficient condition

This section is devoted to the proof of Theorem 7, which states that in the weak dissipation regime $s \in (0,1]$, the fractional heat equation (E_s) is exactly null-controllable from any α-exponentially thick set $\omega \subset \mathbb{R}^n$, with $0 < \alpha < s$, and in any positive time $T > 0$. This done in two steps: first proving a spectral estimate reminiscent of Jerison and Lebeau's spectral inequality [8, Theorem 14.6] or Logvinenko-Sereda-Kovrijkine estimate [10], and second using Lebeau and Robbiano's method, as stated in the following theorem that was proved by Miller [14, Theorem 2.2] (see also [15, Theorem 2.8]).

[1] I.e., if $I_{nk} = [a_{nk}, b_{nk}]$, set $b'_{nk} := (a_{nk}(1+r_n) + b_{nk}(1-r_n))/2$ and $a'_{nk} = (a_{nk}(1-r_n) + b_{nk}(1+r_n))/2$, and finally $I'_{nk} := [a_{nk}, b'_{nk}] \cup [a'_{nk}, b_{nk}]$.

Theorem 11. *Let A be a non-negative selfadjoint operator on $L^2(\mathbb{R}^n)$, and let $\omega \subset \mathbb{R}^d$ be measurable. Suppose that there are $d_0 > 0$, $d_1 \geq 0$, and $\zeta \in (0,1)$ such that for all $\lambda \geq 0$ and $f \in \operatorname{Ran} \mathbb{1}_{(-\infty,\lambda]}(A)$,*

$$\|f\|_{L^2(\mathbb{R}^n)} \leq d_0 e^{d_1 \lambda^\zeta} \|f\|_{L^2(\omega)}.$$

Then, there exist positive constants $c_1, c_2, c_3 > 0$, only depending on ζ, such that for all $T > 0$ and $g \in L^2(\mathbb{R}^n)$ we have the observability estimate

$$\|e^{-TA} g\|^2_{L^2(\mathbb{R}^n)} \leq \frac{C_{obs}}{T} \int_0^T \|e^{-tA} g\|^2_{L^2(\omega)} \, dt,$$

where the positive constant $C_{obs} > 0$ is given by

$$C_{obs} = c_1 d_0 (2d_0 + 1)^{c_2} \exp\left(c_3 \left(\frac{d_1}{T^\zeta} \right)^{\frac{1}{1-\zeta}} \right).$$

In the rest of this section, in order to alleviate the text, we will denote the spectral subspaces associated with the Laplacian as follows

$$\mathcal{E}_\lambda = \operatorname{Ran} \mathbb{1}_{(-\infty,\lambda]}(-\Delta), \quad \lambda \geq 0.$$

Let us now prove the following spectral estimates.

Proposition 12. *Let $\alpha \in (0,1)$ and let $\omega \subset \mathbb{R}^n$ be a measurable set that is α-exponentially thick. Then, there exists a positive constant $c > 0$ such that*

$$\forall \lambda > 0, \forall f \in \mathcal{E}_\lambda, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq c e^{c\lambda^{\alpha/2}} \|f\|_{L^2(\omega)}.$$

Remark 13. Recall from Kovrikine's estimate [10, Theorem 3] that there exists a universal positive constant $K > 0$ depending only on the dimension n such that for all (γ, L)-thick set $\omega \subset \mathbb{R}^n$, with $\gamma \in (0,1]$ and $L > 0$, we have

$$\forall \lambda \geq 0, \forall f \in \mathcal{E}_\lambda, \quad \|f\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K}{\gamma} \right)^{K(1+L\sqrt{\lambda})} \|f\|_{L^2(\omega)}. \tag{1}$$

Our spectral estimates therefore improve the ones known for the thick sets.

Proof of proposition 12. Let us consider some $\lambda > 0$ and $f \in \mathcal{E}_\lambda$ be fixed. Since an α-exponentially thick set is $ce^{-CL^{-\alpha}}$-thick at every scale $0 < L < L_0$, for some $c > 0$, $C > 0$ and $L_0 > 0$, we deduce from (1) that

$$\forall L \in (0, L_0), \quad \|f\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K}{ce^{-CL^{-\alpha}}} \right)^{K(1+L\sqrt{\lambda})} \|f\|_{L^2(\omega)}.$$

Assume for a moment that $\lambda \geq 1$. Then, by choosing $L = L_0/\sqrt{\lambda}$ in the above estimate, we obtain that

$$\|f\|_{L^2(\mathbb{R}^n)} \leq \left(\frac{K}{c} \right)^{K(1+L_0)} e^{CK(1+L_0)L_0^{-\alpha}\lambda^{\alpha/2}} \|f\|_{L^2(\omega)}.$$

This is the expected estimate when $\lambda \geq 1$. For the case $0 < \lambda < 1$, we use again Kovrijkine's estimate to find a C_1 such that for every $f \in \mathcal{E}_\lambda \subset \mathcal{E}_1$, $\|f\|_{L^2(\mathbb{R}^n)} \leq C_1 \|f\|_{L^2(\omega)}$. Therefore, for any $c_0 > 0$, and in particular for $c_0 = CL(1 + L_0)L_0^{-\alpha}$

$$\|f\|_{L^2(\mathbb{R}^n)} \leq C_1 e^{c_0 \lambda^{\alpha/2}} \|f\|_{L^2(\omega)}.$$

This ends the proof of proposition 12. □

Proof of theorem 7. Given a positive time $T > 0$ and a α-exponentially thick set $\omega \subset \mathbb{R}^n$, with $0 < \alpha < s$, we are now in position to prove an exact observability estimate for the fractional heat semigroup from ω at time T. First notice that

$$\mathbb{1}_{(-\infty,\lambda]}((-\Delta)^{s/2}) = \mathbb{1}_{(-\infty,\lambda^{2/s}]}(-\Delta), \quad \lambda \geq 0.$$

We therefore deduce from Proposition 12 that there exists a positive constant $c > 0$ such that

$$\forall \lambda > 0, \forall f \in \operatorname{Ran} \mathbb{1}_{(-\infty,\lambda]}((-\Delta)^{s/2}), \quad \|f\|_{L^2(\mathbb{R}^n)} \leq c e^{c\lambda^{\alpha/s}} \|f\|_{L^2(\omega)}.$$

Theorem 11 then implies that there exists a positive constant $C > 0$ such that for all $T > 0$ and $g \in L^2(\mathbb{R}^n)$,

$$\|e^{-T(-\Delta)^{s/2}} g\|^2_{L^2(\mathbb{R}^n)} \leq C \exp\left(\frac{C}{T^{\alpha/(s-\alpha)}}\right) \frac{1}{T} \int_0^T \|e^{-t(-\Delta)^{s/2}} g\|^2_{L^2(\omega)} \, dt.$$

This is the expected estimate. □

4. NECESSARY CONDITION

The aim of this section is to prove Theorem 5. This necessary condition is proved by testing the observability inequality on coherent states, as in [9].

Proof of theorem 5. We detail the proof in dimension 1. The proof in higher dimension is very similar. Indeed, all the tools and theorems we use can be adapted in higher dimension (see [9, §4.3]).

Step 1: Observability inequality. — As in the proof of theorem 7 (and see [4, Theorem 2.44]), the exact null-controllability of the fractional heat equation (E_s) on ω in time T is equivalent to the following observability inequality: for every $g_0 \in L^2(\mathbb{R})$, the solution g of $(\partial_t + (-\Delta)^{s/2})g(t,x) = 0$, $g(0,\cdot) = g_0$ satisfies

$$\|g(T,\cdot)\|_{L^2(\mathbb{R})} \leq C\|g\|_{L^2((0,T)\times\omega)}. \tag{2}$$

Throughout this proof, c and C denote constants that can change from line to line.

Step 2: Construction of test functions. — We want to find a lower bound on $\mathrm{Leb}(\omega \cap B(x,L))$. Since the fractional heat equation is invariant by translation, we may assume that $x = 0$.

Let $\xi_0 > 0$ and $\chi \in C_c^\infty(-\xi_0, \xi_0)$ such that $\chi \equiv 1$ on a neighborhood of 0. For $h > 0$, set

$$\begin{aligned} g_h(t,x) &:= h \int_{\mathbb{R}} \chi(h\xi - \xi_0) e^{-(h\xi-\xi_0)^2/2h + ix\xi - t|\xi|^s} \, d\xi \\ &:= \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-(\xi-\xi_0)^2/2h + ix\xi/h - t|\xi|^s/h^{-s}} \, d\xi. \end{aligned} \tag{3}$$

Then g_h is a solution to the fraction heat equation (E_s).

Step 3: Estimates g_h. — With the notations of [9, Section 3.2], we have $g_h(t,x) = I_{t,h,1}(x)$ with $\rho_t(\xi) = -t\xi^s$. According to [9, Proposition 3.5] (with $X = [0,T]$), there exists $\eta > 0$ such that uniformly in $0 \leq t \leq T$ and $|x| < \eta$,

$$g_h(t,x) = \sqrt{2\pi h}\, e^{ix\xi_0/h - x^2/2h - t(\xi_0+ix)^s h^{-s} + O(h^{1-2s})}(1 + O(h^{1-s})). \tag{4}$$

Hence,

$$\|g_h(T,\cdot)\|_{L^2(\mathbb{R})} \geq \|g_h(T,\cdot)\|_{L^2(|x|<\eta)} \geq c e^{-Ch^{-s}}. \tag{5}$$

Moreover, according to [9, Proposition 3.7] (with $X = [0,T]$), uniformly in $0 \leq t \leq T$ and $|x| > \eta$

$$g_h(t,x) = O(|x|^{-2} e^{-c/h}). \tag{6}$$

Hence, for every $h > 0$ small enough

$$\begin{aligned}\|g_h\|^2_{L^2((0,T)\times\omega)} &= \|g_h\|^2_{L^2((0,T)\times\{\eta<|x|\}\cap\omega)} + \|g_h\|^2_{L^2((0,T)\times\{\eta<|x|\}\cap\omega)} \\ &\leq C\mathrm{e}^{-c/h} + \|g_h\|^2_{L^2((0,T)\times\{|x|<\eta\}\cap\omega)}.\end{aligned}$$

Now, set $\beta := (1-s)/2$. According to the asymptotics (4), there exist $C, c > 0$ such that for every $r > 0$ and every $h > 0$ small enough so that $rh^\beta < \eta$,

$$\begin{aligned}\|g_h\|^2_{L^2((0,T)\times\omega)} &\leq C\mathrm{e}^{-c/h} + \|g_h\|^2_{L^2(0,T)\times\{rh^\beta<|x|<\eta\})} + \|g_h\|^2_{L^2(0,T)\times\{|x|<rh^\beta\})} \\ &\leq C\mathrm{e}^{-c/h} + Ch\mathrm{e}^{-cr^2h^{2\beta-1}+O(h^{1-2s})} + Ch\operatorname{Leb}(\omega\cap B(0,rh^\beta)).\end{aligned}$$

Since $2\beta - 1 = -s$ and $1-2s > -s$, there exist $C, c > 0$ such that for every $r > 0$ and every $h > 0$ small enough (depending on r),

$$\|g_h\|^2_{L^2((0,T)\times\omega)} \leq C\mathrm{e}^{-cr^2h^{-s}} + Ch\operatorname{Leb}(\omega\cap B(0,rh^\beta)). \tag{7}$$

Step 4: Conclusion. — If the observability inequality (2) holds, according to eqs. (5) and (7), there exist $c, C > 0$ such that for any $r > 0$ and every $h > 0$ small enough (depending on r):

$$c\mathrm{e}^{-Ch^{-s}} \leq C\mathrm{e}^{-cr^2h^{-s}} + Ch\operatorname{Leb}(\omega\cap B(0,rh^\beta)).$$

If we choose r large enough, we can absorb the $\mathrm{e}^{-cr^2h^{-s}}$ in the left-hand side. Hence, for every $h > 0$ small enough,

$$\mathrm{e}^{-Ch^{-s}} \leq C\operatorname{Leb}(\omega\cap B(0,rh^\beta)).$$

Setting $L = rh^\beta$, i.e., $h = cL^{1/\beta}$,

$$\operatorname{Leb}(\omega\cap B(0,L)) \geq c\mathrm{e}^{-CL^{-s/\beta}} = c\mathrm{e}^{-CL^{-2s/(1-s)}}. \qquad \square$$

5. Examples of exponentially-thick sets

Let us first recall some basic facts about Smith-Volterra-Cantor sets. At each step in the construction of a Smith-Volterra-Cantor set, we remove a subset of measure $(1-r_n)\operatorname{Leb}(K_n)$ from K_n (see definition 9), hence:

Proposition 14. *With the notations of definition 9,* $\operatorname{Leb}(K_n) = \prod_{k=0}^{n-1}(1-r_n)$ *and* $\operatorname{Leb}(K) = \prod_{n=0}^{+\infty}(1-r_n)$. *In particular,* $\operatorname{Leb}(K) > 0$ *if and only if* $\sum_n r_n < +\infty$.

Our first result is a some upper and lower bounds on the thickness of the complement of Smith-Volterra-Cantor sets.

Proposition 15. *Let* $(r_n)_n \in (0,1)^{\mathbb{N}}$. *Assume that* $\sum_n r_n < +\infty$. *Let K be the associated Smith-Volterra-Cantor set. Set* $c_0 := \operatorname{Leb}(K)$ *and* $\omega := \mathbb{R}\setminus K$. *There exist $c > 0$, $C > 0$ and $L_0 > 0$ such that for every* $0 < L < L_0$,

$$c\sum_{k\geq\log_2(3c_0/L)} r_k \leq \inf_{x\in\mathbb{R}} \frac{\operatorname{Leb}(\omega\cap B(x,L))}{\operatorname{Leb}(B(x,L))} \leq C\sum_{k\geq\log_2(c_0/4L)} r_k,$$

where $\log_2$ *is the base 2 logarithm* $\log_2(x) = \ln(x)/\ln(2)$.

With a more careful analysis in the proof below, it seems we could improve this inequality by replacing the $\log_2(3c_0/L)$ by $\log_2(\varkappa c_0/L)$ with some $\varkappa < 3$. We don't know what the optimal $\varkappa$ is. We don't pursue this because we don't need such a sharp estimate. In fact proposition 15 is already sharper than we need for our purposes.

Proof. Remark that we only need to estimate $\mathrm{Leb}(\omega \cap B(x, L))$ for $x \in [0, 1]$. Indeed, if for instance $x > 1$, $\omega \cap B(x, L)$ contains at least $[x, x + L]$, hence $\mathrm{Leb}(\omega \cap B(x, L))/\mathrm{Leb}(B(x, L)) \in (1/2, 1)$.

Step 1: Notations and preliminary computations. — In this proof, we denote by I_{nk} the intervals that appears in the construction of K, as defined in definition 9. We denote the length of I_{nk} (which does not depend on k) by ℓ_n. We have

$$\ell_n = \frac{1 - r_{n-1}}{2}\ell_{n-1}. \tag{8}$$

Notice that

$$\mathrm{Leb}(I_{nk} \cap \omega) = \mathrm{Leb}(I_{nk}) - \mathrm{Leb}(I_{nk} \cap K) = \ell_n\big(1 - \prod_{k\geq n}(1 - r_k)\big).$$

In addition, we can estimate the right-hand side in the following way

$$\begin{aligned}
1 - \prod_{k\geq n}(1 - r_k) &= 1 - \exp\Big(\sum_{k\geq n} \ln(1 - r_k)\Big)\\
&= 1 - \exp\Big(\sum_{k\geq n} -r_k(1 + o_k(1))\Big)\\
&= 1 - \exp\Big(-(1 + o_n(1))\sum_{k\geq n} r_k\Big)\\
&= 1 - \Big(1 - (1 + o_n(1))\sum_{k\geq n} r_k\Big)\\
&= (1 + o_n(1))\sum_{k\geq n} r_k.
\end{aligned}$$

In the third and fourth relations, we used that $\sum_k r_k < +\infty$. Thus,

$$\mathrm{Leb}(I_{nk} \cap \omega) = (1 + o_n(1))\ell_n \sum_{k\geq n} r_k. \tag{9}$$

Step 2: Lower bound when L is comparable to ℓ_n. — Let $L > 0$ and $n \in \mathbb{N}$ be such that $2\ell_n \leq L \leq 6\ell_n$. Let $x \in [0, 1]$.

If $\omega \cap B(x, L)$ contains an interval of length $\geq \ell_n/2$, $\mathrm{Leb}(\omega \cap B(x, L)) \geq \ell_n/2$.

If that is not the case, then, $\mathrm{distance}(x, K_n) < \ell_n/4$. Since $L \geq 2\ell_n$, this implies that $B(x, L)$ contains some I_{nk}. Hence, according to eq. (9),

$$\mathrm{Leb}(\omega \cap B(x, L)) \geq \mathrm{Leb}(\omega \cap I_{nk}) = (1 + o_n(1))\ell_n \sum_{k\geq n} r_k.$$

Putting the two cases together:

$$\inf_{x\in\mathbb{R}} \frac{\mathrm{Leb}(\omega \cap B(x, L))}{\mathrm{Leb}(B(x, L))} \geq \frac{\ell_n}{2L}\min\Big(c\sum_{k\geq n} r_k, \frac{1}{2}\Big) \geq \frac{c\ell_n}{2L}\sum_{k\geq n} r_k.$$

Since $L \leq 6\ell_n$,

$$\inf_{x\in\mathbb{R}} \frac{\mathrm{Leb}(\omega \cap B(x, L))}{\mathrm{Leb}(B(x, L))} \geq c\sum_{k\geq n} r_k, \tag{10}$$

this inequality being valid whenever $2\ell_n \leq L \leq 6\ell_n$.

Step 3: Upper bound when L is comparable to ℓ_n. — Let $L > 0$ and $n \in \mathbb{N}$ be such that $\ell_n/3 \leq 2L \leq \ell_n$. Let $x \in [0, 1]$ in the middle of a I_{nk}, so that $B(x, \ell_n/2) = I_{nk}$. Then $B(x, L) \subset I_{nk}$, and according to eq. (9),

$$\mathrm{Leb}(\omega \cap B(x, L)) \leq \mathrm{Leb}(I_{nk} \cap \omega) = (1 + o_n(1))\ell_n \sum_{k\geq n} r_k.$$

Since, $\ell_n/3 \le 2L$,

$$\inf_{x\in\mathbb{R}} \frac{\mathrm{Leb}(\omega\cap B(x,L))}{\mathrm{Leb}(B(x,L))} \le C\sum_{k\ge n} r_k, \tag{11}$$

this inequality being valid whenever $\ell_n/3 \le 2L \le \ell_n$.

Step 4: Solving the inequality $a\ell_n \le L \le b\ell_n$. — Let $L>0$ and $0<\varkappa<1$. Set $n(L) := \lceil \log_2(c_0/L)\rceil$, where $\lceil\cdot\rceil$ is the ceiling function. We aim to prove that for L small enough, $\varkappa\ell_{n(L)} \le L \le 2\ell_{n(L)}$.

According to the definition of I_{nk}, $\ell_n = 2^{-n}\prod_{k=0}^{n-1}(1-r_k)$. Recall that $c_0 = \prod_{k=0}^{+\infty}(1-r_k) > 0$. Define ϵ_n by $1+\epsilon_n = \big(\prod_{k\ge n}(1-r_k)\big)^{-1}$. Then $\epsilon_n > 0$ and $\epsilon_n \to 0$ as $n\to+\infty$. With this notation, we have the equivalences

$$\begin{aligned}\varkappa\ell_n \le L \le 2\ell_n &\Leftrightarrow 2^{-n}\varkappa c_0(1+\epsilon_n) \le L \le 2^{1-n}c_0(1+\epsilon_n)\\ &\Leftrightarrow -n+\log_2(\varkappa c_0(1+\epsilon_n)) \le \log_2(L) \le 1-n+\log_2(c_0(1+\epsilon_n))\\ &\Leftrightarrow \log_2\left(\frac{c_0}{L}\right)+\log_2(\varkappa)+\log_2(1+\epsilon_n) \le n \le \log_2\left(\frac{c_0}{L}\right)+1+\log_2(1+\epsilon_n).\end{aligned} \tag{12}$$

According to the definition of the ceiling function,

$$\log_2\left(\frac{c_0}{L}\right) \le n(L) < \log_2\left(\frac{c_0}{L}\right)+1.$$

Since $\log_2(\varkappa)<0$, $\log_2(1+\epsilon_n)>0$ and $\epsilon_n\to 0$, the inequalities (12) are satisfied for $n=n(L)$ and small enough $L>0$.

Step 5: Conclusion. — Applying the previous step with $\varkappa = 2/3$, and replacing L by $L/3$, we see that lower bound (10) holds for $n = \lceil \log_2\left(\frac{3c_0}{L}\right)\rceil$ when L is small enough. Hence, the lower-bound stated in proposition 15 holds.

Applying Step 4 with $\varkappa = 2/3$ and L replaced by $4L$, we get that the upper bound (11) holds with $n = \lceil \log_2\left(\frac{c_0}{4L}\right)\rceil$ and L small enough, which gives the stated upper bound. □

Applying this general bound, we now prove that with the proper sequence $(r_n)_n$, the complement of the associated Smith-Volterra-Cantor set is α-exponentially thick.

Proof of Theorem 10. Since $r_n = c\exp(-C2^{n\alpha})$ decays faster than exponentially, $\sum_n r_n < +\infty$. Moreover, as $n\to+\infty$

$$\sum_{k\ge n} r_k = r_n(1+o(1)).$$

Plugging this into the bounds of proposition 15 proves the claimed bounds. □

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References

[1] P. Alphonse & J. Bernier, *Smoothing properties of fractional Ornstein-Uhlenbeck semigroups and null-controllability*, Bull. Sci. Math. **165** (2020), 102914, 52 pp.

[2] P. Alphonse & J. Martin, *Stabilization and approximate null-controllability for a large class of diffusive equations from thick control supports*, ESAIM, Control Optim. Calc. Var. **28** (2022), 30 pp.

[3] K. Beauchard, M. Egidi & K. Pravda-Starov, *Geometric conditions for the null-controllability of hypoelliptic quadratic parabolic equations with moving control supports*, C. R. Math. Acad. Sci. Paris **358** (2020), 651–700.

[4] J.-M. Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs, Vol. **136**, American Mathematical Society, Providence, RI (2007).

[5] M. Egidi & A.Seelmann, *An abstract Logvinenko-Sereda type theorem for spectral subspaces*, J. Math. Anal. Appl. **500** (2021), 125149.

[6] M. Egidi & I. Veselić, *Sharp geometric condition for null-controllability of the heat equation on $\mathbb{R}^d$ and consistent estimates on the control cost*, Arch. Math. **111** (2018), 85–99.

[7] S. HUANG, G. WANG & M. WANG, *Characterizations of stabilizable sets for some parabolic equations in* $\mathbb{R}^n$, J. Differential Equations **272** (2021), 255–288.
[8] D. JERISON & G. LEBEAU, *Nodal Sets of Sums of Eigenfunctions*, in: Harmonic analysis and partial differential equations. Essays in honor of Alberto P. Calderón's 75th birthday. Proceedings of a conference, University of Chicago, IL, USA, February 1996. Chicago, IL: The University of Chicago Press. 223–239 (1999).
[9] A. KOENIG, *Lack of null-Controllability for the fractional heat equation and related equations*, SIAM J. Control Optim. **58** (2020), 3130–3160.
[10] O. KOVRIJKINE, *Some results related to the Logvinenko-Sereda Theorem*, Proc. Amer. Math. Soc. **129**, (2001), 3037–3047.
[11] G. LEBEAU & L. ROBBIANO, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), 335–356.
[12] P. LISSY, *A non-controllability result for the half-heat equation on the whole line based on the prolate spheroidal wave functions and its application to the Grushin equation*, preprint (2022), hal-02420212.
[13] H. LIU, G. WANG, Y. XU & H. YU, *Characterizations on complete stabilizability*, SIAM J. Control Optim. **60** (2022), 2040–2069.
[14] L. MILLER, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups*, Discrete Contin. Dyn. Syst. Ser. B **14** (2010), 1465–1485.
[15] I. NAKIĆ, M. TÄUFER, M. TAUTENHAN & I. VESELIĆ, *Sharp estimates and homogenization of the control cost of the heat equation on large domains*, ESAIM, Control Optim. Calc. Var. **26** (2020), 26 pp.
[16] E. TRÉLAT, G. WANG & Y. XU, *Characterization by observability inequalities of controllability and stabilization properties*, Pure Appl. Anal. **2** (2020), no. 1, pp. 93-122.
[17] G. WANG, M. WANG, C. ZHANG & Y. ZHANG, *Observable set, observability, interpolation inequality and spectral inequality for the heat equation in* $\mathbb{R}^n$, J. Math. Pures Appl. (9) **126** (2019), 144–194.

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