J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1307

Published by EMS Press

Jean-Michel Coron · Armand Koenig · Hoai-Minh Nguyen

On the small-time local controllability of a KdV system for critical lengths

Received October 9, 2020

Abstract. This paper is devoted to the local null-controllability of the nonlinear KdV equation equipped the Dirichlet boundary conditions using the Neumann boundary control on the right. Rosier proved that this KdV system is small-time locally controllable for all noncritical lengths and that the uncontrollable space of the linearized system is of finite dimension when the length is critical. Concerning critical lengths, Coron and Crépeau showed that the same result holds when the uncontrollable space of the linearized system is of dimension 1; later Cerpa, and then Cerpa and Crépeau, established that the local controllability holds at a finite time for all other critical lengths. In this paper, we prove that, for a class of critical lengths, the nonlinear KdV system is *not* small-time locally controllable.

Keywords. Controllability, nonlinearity, KdV equations, power series expansion, Hilbert uniqueness method

Contents

Jean-Michel Coron: Laboratoire Jacques-Louis Lions, Sorbonne Université, Université de Paris, CNRS, INRIA, Paris, France; coron@ann.jussieu.fr

Armand Koenig: Laboratoire Jean Alexandre Dieudonné, Université Côte d'Azur, CNRS, Nice, France; armand.koenig@univ-cotedazur.fr

Hoai-Minh Nguyen: Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris, France; hoai-minh.nguyen@sorbonne-universite.fr

Mathematics Subject Classification (2020): Primary 93B05; Secondary 93C15, 76B15

1. Introduction

We are concerned with the local null-controllability of the (nonlinear) KdV equation equipped the Dirichlet boundary conditions using the Neumann boundary control on the right. More precisely, given $L > 0$ and $T > 0$, we consider the following control system:

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) + y(t,x)y_x(t,x) = 0 & \text{for } t \in (0,T),\ x \in (0,L), \\ y(t, x = 0) = y(t, x = L) = 0 & \text{for } t \in (0,T), \\ y_x(t, x = L) = u(t) & \text{for } t \in (0,T), \end{cases} \tag{1.1}$$

and

$$y(t = 0, x) = y_0(x) \quad \text{for } x \in (0, L). \tag{1.2}$$

Here y is the state, y_0 is the initial datum, and u is the control. More precisely, we are interested in the *small-time local controllability* of this system.

The KdV equation has been introduced by Boussinesq [15] and Korteweg and de Vries [30] as a model for propagation of surface water waves along a channel. This equation also furnishes a very useful nonlinear approximation model including a balance between weak nonlinearity and weak dispersive effects. The KdV equation has been intensively studied from various aspects, including well-posedness, existence and stability of solitary waves, integrability, long-time behavior, etc.; see e.g. [29, 31, 33, 44, 46].

1.1. History

The controllability properties of system (1.1)–(1.2) (or of its variants) have been studied intensively; see e.g. the surveys [19, 40] and the references therein. Let us briefly review the existing results on (1.1) and (1.2). For the initial and final datum in $L^2(0, L)$ and controls in $L^2(0, T)$, Rosier [38] proved that the system is small-time locally controllable around 0 provided that the length L is not critical, i.e., $L \notin \mathcal{N}$, where

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}};\ k, l \in \mathbb{N}_* \right\}. \tag{1.3}$$

To this end, he studied the controllability of the linearized system using the Hilbert Uniqueness Method and compactness-uniqueness arguments. Rosier also showed that the linearized system is controllable if $L \notin \mathcal{N}$. Moreover, he established that when $L \in \mathcal{N}$, the linearized system is not controllable. More precisely, he showed that there exists a nontrivial finite-dimensional subspace $\mathcal{M}$ of $L^2(0, L)$ such that its orthogonal space is reachable from 0 whereas $\mathcal{M}$ is not.

To tackle the control problem for the critical length $L \in \mathcal{N}$ with initial and final datum in $L^2(0, L)$ and controls in $L^2(0, T)$, Coron and Crépeau introduced the power series expansion method [24]. The idea is to take into account the effect of the nonlinear term yy_x absent in the linearized system. Using this method, they showed [24] (see also [22, Section 8.2]) that system (1.1)–(1.2) is small-time locally controllable if $L = m2\pi$ for $m \in \mathbb{N}_*$ satisfying

$$\nexists (k, l) \in \mathbb{N}_* \times \mathbb{N}_* \text{ with } k^2 + kl + l^2 = 3m^2 \text{ and } k \neq l. \tag{1.4}$$

In this case, $\dim \mathcal{M} = 1$ and $\mathcal{M}$ is spanned by $1 - \cos x$. Cerpa [18] developed the analysis in [24] to prove that system (1.1)–(1.2) is locally controllable in *a finite time* in the case $\dim \mathcal{M} = 2$. This corresponds to the case where

$$L = 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}$$

for some $k, l \in \mathbb{N}_*$ with $k > l$, and there are no $m, n \in N_*$ with $m > n$ and $m^2 + mn + n^2 = k^2 + kl + l^2$. Later, Crépeau and Cerpa [20] succeeded in extending the ideas in [18] to obtain local controllability for all other critical lengths in a finite time. To summarize, concerning the critical lengths with initial and final datum in $L^2(0, L)$ and controls in $L^2(0, T)$, small-time local controllability is valid when $\dim \mathcal{M} = 1$ and local controllability in large enough time holds when $\dim \mathcal{M} \geq 2$.

1.2. Statement of the result

The control properties of KdV equations have been intensively studied previously but the following natural question remains open (see [23, Open problem 10], [18, Remark 1.7]):

Open Problem 1.1. Is system (1.1)–(1.2) small-time locally controllable for all $L \in \mathcal{N}$?

In this paper we give a negative answer to this question. We show that system (1.1)–(1.2) is not small-time locally controllable for a class of critical lengths. More precisely, we have

Theorem 1.2. *Let k, $l \in \mathbb{N}_*$ be such that $2k + l \notin 3\mathbb{N}_*$. Assume that*

$$L = 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}.$$

Then system (1.1)–(1.2) is not small-time locally null-controllable with controls in H^1 and initial and final datum in $H^3(0, L) \cap H_0^1(0, L)$, i.e., there exist $T_0 > 0$ and $\varepsilon_0 > 0$ such that, for all $\delta > 0$, there is $y_0 \in H^3(0, L) \cap H_0^1(0, L)$ with $\|y_0\|_{H^3(0,L)} < \delta$ such that for all $u \in H^1(0, T_0)$ with $\|u\|_{H^1(0,T_0)} < \varepsilon_0$ and $u(0) = y_0'(L)$, we have

$$y(T_0, \cdot) \not\equiv 0,$$

where $y \in C([0, T_0]; H^3(0, L)) \cap L^2([0, T_0]; H^4(0, L))$ is the unique solution of (1.1)–(1.2).

Open Problem 1.3. We have not been able to establish that the control system (1.1)–(1.2) is not small-time locally controllable with initial and final datum in $L^2(0, L)$ and control in $L^2(0, T)$ for a critical length as in Theorem 1.2. It would be interesting to extend the method in the paper to deal with this problem. It would also be interesting to know what is the smallest s such that system (1.1)–(1.2) is not small-time locally controllable with controls in $H^s(0, T)$, and initial and final datum in $D(\mathcal{A}^s)$, $\mathcal{A}$ being defined in Lemma 2.1 below.

Remark 1.4. Concerning Open Problem 1.3, maybe the smallest s is not an integer, as in the nonlinear parabolic equation studied in [8], a phenomenon which is specific to the infinite dimension as shown in [7]. Note that in [32] a noninteger s already appears for an obstruction to small-time local controllability; however, it is not known if this s is the optimal one.

Open Problem 1.5. It would also be interesting to know what is the optimal time for local null controllability. In particular, one may ask if for $T \le T^{>}$, with $T^{>}$ defined in [20, p. 463], the control system (1.1)–(1.2) is not locally null controllable in time T (for example with initial and final datum in $H^3(0, L) \cap H^1_0(0, L)$ and control in $H^1(0, T)$) for critical lengths L as in the above theorem.

Open Problem 1.6. Finally, it would be interesting to know if the assumption $2k + l \not\in 3\mathbb{N}_*$ can be replaced by the weaker assumption $\dim \mathcal{M} > 1$. In other words, is it true that the control system (1.1)–(1.2) is not small-time locally controllable when $\dim \mathcal{M} > 1$?

In Theorem 1.2, we deal with controls in $H^1(0, T_0)$ and initial and final datum in $H^3(0, L) \cap H^1_0(0, L)$, instead of controls in $L^2(0, T_0)$ and initial and final datum in $L^2(0, L)$ as considered in [18, 20, 24, 38]. For a subclass of critical lengths considered in Theorem 1.2, we prove later (see Theorem 6.1) that system (1.1)–(1.2) is locally controllable with initial and final datum in $H^3(0, L) \cap H^1_0(0, L)$ and controls in $H^1(0, T)$. It is worth noting that even though the propagation speed of the KdV equation is infinite, some time is needed to reach the zero state.

We emphasize that there are other types of boundary controls for the KdV equations for which there is no critical length: see [19, 28, 38, 39]. There are also results on internal controllability for KdV equations: see [42], [17] and references therein.

A minimal time of null-controllability is also required for some linear partial differential equations. This is obviously the case for equations with finite speed of propagation, such as the transport equation [22, Theorem. 2.6], or the wave equation [3, 16], or hyperbolic systems [25]. But this can also happen for equations with infinite speed of propagation, such as some parabolic systems [2, 11], Grushin-type equations [4, 9, 26], Kolmogorov-type equations [5] or parabolic-transport coupled systems [6] (see also the references in those papers). Nevertheless, to our knowledge, a minimal time required for KdV equations using boundary controls is established for the first time in this work. This fact is surprising when compared with known results on internal controls for the KdV system (1.1) with $u = 0$. It is known (see [17, 36, 37]) that the KdV system (1.1) with

$u = 0$ is locally controllable using internal controls whenever the control region contains an *arbitrary* open subset of $(0, L)$.

However, our obstruction to small-time local controllability of our KdV control system is of a different nature than these obstructions to small-time null controllability for linear partial differential equations. It comes from a phenomenon which already appears in finite dimensions for *nonlinear* control systems. Note that in finite dimensions, in contrast to the case of partial differential equations as just pointed above, a linear control system which is controllable in large time is controllable in arbitrarily small time. This is no longer the case for nonlinear control systems in finite dimensions: There are nonlinear control systems in finite dimensions which are locally controllable in large enough time but are not locally controllable in small time. A typical example is the control system

$$\dot{y}_1 = u, \quad \dot{y}_2 = y_3, \quad \dot{y}_3 = -y_2 + 2y_1 u, \tag{1.5}$$

where the state is $(y_1, y_2, y_3)^\mathsf{T} \in \mathbb{R}^3$ and the control is $u \in \mathbb{R}$. There are many powerful necessary conditions for small-time local controllability of nonlinear control systems in finite dimensions. Let us mention in particular the Sussmann condition [43, Proposition 6.3]. See also [7] by Beauchard and Marbach for further results, in particular for controls in the Sobolev spaces $H^k(0, T)$, and a different approach. The Sussmann condition [43, Proposition 6.3] tells us that the nonlinear control system (1.5) is not small-time locally controllable (see [22, Example 3.38]): it gives a precise direction, given by an explicit iterated Lie bracket, in which one cannot move in small time. For partial differential equations iterated Lie brackets can sometimes be defined, at least heuristically, for interior controls but are not well understood for boundary controls (see [22, Chapter 5]), which is the type of controls considered here. However, for the simple control system (1.5), an obstruction to small-time local controllability can be obtained by pointing out that if $(y, u) : [0, T] \to \mathbb{R}^3 \times \mathbb{R}$ is a trajectory of the control system (1.5) such that $y(0) = 0$, then

$$y_2(T) = \int_0^T \cos(T - t) y_1(t)^2 \, dt, \tag{1.6}$$

$$y_3(T) = y_1(T)^2 - \int_0^T \sin(T - t) y_1(t)^2 \, dt. \tag{1.7}$$

Hence,

$$y_2(T) \geq 0 \quad \text{if } T \in [0, \pi/2], \tag{1.8}$$

$$y_3(T) \leq 0 \quad \text{if } T \in [0, \pi] \text{ and } y_1(T) = 0, \tag{1.9}$$

which also shows that the control system (1.5) is not small-time locally controllable and more precisely, using (1.9), is not locally controllable in time $T \in [0, \pi]$ ((1.8) gives only an obstruction for $T \in [0, \pi/2]$). Note that condition (1.8), at least for $T > 0$ small enough, is the obstruction to small-time local controllability given by [43, Proposition 6.3], while (1.9) is not related to this proposition. For the control system (1.5) one knows that it is locally controllable in large enough time and the optimal time for local controllability is also known: this control system is locally controllable in time T if and only if $T > \pi$; see [22, Example 6.4]. Moreover, if there are higher order perturbations (with respect to the weight $(r_1, r_2, r_3) = (1, 2, 2)$ for the state and 1 for the control; see [22, Section 12.3]) one can still get an obstruction to small-time local controllability by pointing out that (1.6) and (1.7) respectively imply

$$\text{for every } T \in (0, \pi/2) \text{ there exists } \delta > 0 \text{ such that } y_2(T) \geq \delta |u|^2_{H^{-1}(0,T)}, \tag{1.10}$$

$$\begin{aligned}&\text{for every } T \in (0, \pi] \text{ there exists } \delta > 0 \text{ such that}\\ &\text{if } y_1(T) = 0, \text{ then } y_3(T) \leq -\delta |u|^2_{H^{-2}(0,T)}.\end{aligned} \tag{1.11}$$

Assertion (1.11) follows from the following facts:

$$\int_0^T \Big(\int_0^t y_1(s)\, ds \Big)^2 dt \leq \int_0^T t \int_0^t y_1(s)^2\, ds\, dt \leq T \int_0^T (T - s) y_1(s)^2\, ds,$$
$$\int_0^T \Big(\int_t^T y(s)\, ds \Big)^2 dt \leq \int_0^T (T - s) y(s)^2\, ds,$$

and since $y_1' = u$ and $y_1(0) = 0$,

$$\|u\|^2_{H^{-2}(0,T)} \leq C \int_0^T \Big(\int_0^t y_1(s)\, ds \Big)^2 dt + C \Big(\int_0^T y_1(s)\, ds \Big)^2.$$

Note that (1.10) does not require any condition on the control, while (1.11) requires that the control is such that $y_1(T) = 0$. On the other hand, it is (1.11) that gives the largest time for the obstruction to local controllability in time T: (1.10) gives an obstruction for $T \in [0, \pi/2)$, while (1.11) gives an obstruction for $T \in [0, \pi]$, which in fact is optimal as mentioned above.

There are nonlinear partial differential equations where related inequalities giving an obstruction to small-time local controllability were already proved, namely nonlinear Schrödinger control systems considered by Coron [21] and by Beauchard and Morancey [10], a viscous Burgers equation considered by Marbach [32], and a nonlinear parabolic equation considered by Beauchard and Marbach [8]. Our obstruction to small-time local controllability is also in the same spirit (see in particular Corollary 3.7). Let us briefly explain some of the main ingredients of these previous works.

- In [10, 21], the control is interior and one can compute, at least formally, the iterated Lie bracket [43] in which one could not move in small time (see [22, Section 9.3.1]) if the control systems were in finite dimensions. Then one checks by suitable computations that it is indeed impossible to move in small time in this direction by proving an inequality analogous to (1.11). The computations are rather explicit due to the fact that the drift[1] of the linearized control system is skew-adjoint with explicit and simple eigenvalues and eigenfunctions.

[1] If the linearized control system is written in the form $\dot{y} = Ay + Bu$, the *drift* term is the map $y \mapsto Ay$.

- In [32] the control is again interior. However, the iterated Lie bracket [43] in the direction of which one could not move in small time turns out to be 0. Hence it does not produce any obstruction to small-time local controllability. However, an inequality analogous to (1.10) is proved, but with a fractional (noninteger) Sobolev norm. An important tool of the proof is a change of time-scale which allows one to take an expansion with respect to a new parameter. In the framework of (1.5), this leads to a boundary layer which is analyzed thanks to the maximum principle. Here the drift term of the linearized control system is self-adjoint with explicit and simple eigenvalues and eigenfunctions.
- In [8] the control is again an interior control. Two cases are considered: a case [8, Theorem 3] related to [10, 21] (already analyzed above) and a case [8, Theorem 4] where classical obstructions relying on iterated Lie brackets fail. Concerning [8, Theorem 4] the proof relies on an inequality of type (1.11); its proof can be performed by explicit computations due to some special structure of the quadratic form one wants to analyze: roughly speaking, it corresponds to the case (see [8, (4.17)]) where (3.6) below would be replaced by

$$\int_0^L \int_0^{+\infty} |y(t,x)|^2 \varphi_x(x) e^{-ipt}\, dt\, dx = \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z)} \int_0^L B(z,x)\, dx\, dz, \qquad (1.12)$$

  which simplifies the analysis of the left hand side (in (3.6) one has $\hat{u}(z)\overline{\hat{u}(z-p)}$ instead of $\hat{u}(z)\overline{\hat{u}(z)}$). The computations are also simplified by the fact that the drift term of the linearized control system is self-adjoint with, again, explicit eigenvalues and eigenfunctions.

In this article we prove an estimate of type (1.11), instead of (1.10), expecting that with more precise estimates one might get the optimal time for local controllability as for the control system (1.5). The main differences between our study and the previous articles are the following:

- This is the first case dealing with boundary controls. In our case one does not know what are the iterated Lie brackets even heuristically. Let us take this opportunity to point out that even if they are expected not to live in the state space (see [22, pp. 181–182]), it would be very interesting to understand what are these iterated Lie brackets.
- It seems difficult to perform the change of time-scale introduced in [32] in our situation. Indeed, this change will also lead to a boundary layer. However, one can no longer use the maximum principle to study this boundary layer. Moreover, if the change of time-scale, if justified, allows simpler computations,[2] the advantage of not using it might be to get better or more explicit time for the obstruction to small-time local controllability.
- The linear drift term of the linearized control system (i.e. the operator $\mathcal{A}$ defined in Lemma 2.1) is neither self-adjoint nor skew-adjoint. Moreover, its eigenvalues and eigenfunctions are not explicit.
- Finally, (1.12) does not hold.

[2]This is in particular due to the fact that for the limit problem one again has (1.12).

1.3. Ideas of the analysis

Our approach is inspired by the power series expansion method introduced by Coron and Crépeau [24]. The idea of this method is to search/understand a control u of the form

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots .$$

The corresponding solution then formally has the form

$$y = \varepsilon y_1 + \varepsilon^2 y_2 + \cdots ,$$

and the nonlinear term yy_x can be written as

$$yy_x = \varepsilon^2 y_1 y_{1,x} + \cdots .$$

One then obtains the following systems:

$$\begin{cases} y_{1,t}(t,x) + y_{1,x}(t,x) + y_{1,xxx}(t,x) = 0 & \text{for } t \in (0,T),\ x \in (0,L), \\ y_1(t,x=0) = y_1(t,x=L) = 0 & \text{for } t \in (0,T), \\ y_{1,x}(t,x=L) = u_1(t) & \text{for } t \in (0,T), \end{cases} \tag{1.13}$$

$$\begin{cases} y_{2,t}(t,x) + y_{2,x}(t,x) + y_{2,xxx}(t,x) + y_1(t,x)y_{1,x}(t,x) = 0 & \\ & \text{for } t \in (0,T),\ x \in (0,L), \\ y_2(t,x=0) = y_2(t,x=L) = 0 & \text{for } t \in (0,T), \\ y_{2,x}(t,x=L) = u_2(t) & \text{for } t \in (0,T). \end{cases} \tag{1.14}$$

The idea in [18, 20], with its root in [24], is then to find u_1 and u_2 such that if $y_1(0,\cdot) = y_2(0,\cdot) = 0$, then $y_1(T,\cdot) = 0$ and the $L^2(0,L)$-orthogonal projection of $y_2(T)$ on $\mathcal{M}$ is a given (nonzero) element in $\mathcal{M}$. In [24], the authors needed to make an expansion up to order 3 since y_2 belongs to the orthogonal space of $\mathcal{M}$ in this case. To this end, in [18, 20, 24], the authors used delicate contradiction arguments to capture the structure of KdV systems.

The analysis in this paper has the same roots as the ones mentioned above. Nevertheless, instead of using a contradiction argument, our strategy is to characterize all possible u_1 which steer 0 at time 0 to 0 at time T (see Proposition 2.8). This is done by taking the Fourier transform with respect to time of the solution y_1 and applying Paley–Wiener's theorem. Surprisingly, in the case $2k + l \neq 3\mathbb{N}_*$, if the time T is sufficiently small, there are directions in $\mathcal{M}$ which cannot be reached via y_2 (see Corollary 3.7 and Lemma 5.3). This is one of the crucial observations in this paper. Using this observation, we then implement a method to prove the obstruction to small-time local null-controllability of the KdV system; see Theorem 5.1. The idea is to bring the nonlinear context to the one based on the power series expansion approach, where the new phenomenon is observed (the context of Corollary 3.7). To be able to reach the result of Theorem 1.2, we establish several new estimates for linear and nonlinear KdV systems using low regularity data (see Section 4.2 for the linear and Lemma 5.4 for the nonlinear settings). Their proofs partly involve a connection between linear KdV equations and linear KdV-Burgers equations as previously used by Bona et al. [13] and inspired by the work of Bourgain [14] and Molinet and Ribaud [34]. To establish local controllability for a subclass of critical lengths in a finite time (Theorem 6.1), we again apply the power series method and use a fixed point argument. The key point here is to first obtain controls in $H^1(0, T)$ to control directions which can be reached via the linearized system, and then to obtain controls in $H^1(0, T)$ for y_1 and y_2 mentioned above. The first part is based on a modification of the Hilbert Uniqueness Method, and the second part is again based on the information obtained in Corollary 3.7 and Lemma 5.3. Our fixed point argument is inspired by [18, 24] but is different, somewhat simpler, and, more importantly, relies on the usual Banach fixed point theorem instead of the Brouwer fixed point theorem, which might be interesting for handling nonlinear partial differential equations such that $\mathcal{M}$ is of infinite dimension, as, for example, in [32].

1.4. Structure of the paper

The paper is organized as follows. Section 2 is devoted to the study of controls which steer 0 to 0 (motivated by the system of y_1). In Section 3, we study attainable directions for small time via the power series approach (motivated by the system of y_2). The main result in this section is Proposition 3.6 whose consequence (Corollary 3.7) is crucial to the proof of Theorem 1.2. In Section 4, we establish several useful estimates for linear KdV systems. In Section 5, we give the proof of Theorem 1.2. In fact, we establish a result (Theorem 5.1) which implies Theorem 1.2 and reveals a connection with unreachable directions via the power series expansion method. In Section 6, we establish local controllability for the nonlinear KdV system (1.1) with initial and final datum in $H^3(0, L) \cap H_0^1(0, L)$ and controls in $H^1(0, 1)$ for some critical lengths (Theorem 6.1). In the appendix, we establish various results used in Sections 2 to 4.

2. Properties of controls steering 0 at time 0 to 0 at time T

In this section, we characterize the controls that steer 0 to 0 for the linearized KdV system at a given time. This is done by considering the Fourier transform in the t-variable and the characterization written in terms of Paley–Wiener's conditions. The resolvent of $\partial_x^3 + \partial_x$ hence naturally appears during this analysis. We begin with the discreteness of the spectrum of this operator.

Lemma 2.1. *Set $D(\mathcal{A}) = \{v \in H^3(0, L);\ v(0) = v(L) = v'(L) = 0\}$ and let $\mathcal{A}$ be the unbounded operator on $L^2(0, L)$ with domain $D(\mathcal{A})$ and defined by $\mathcal{A}v = v''' + v'$ for $v \in D(\mathcal{A})$. The spectrum of $\mathcal{A}$ is discrete.*

Proof. Since $\mathcal{A}$ is closed, we only have to prove that there exists a discrete set $\mathcal{D} \subset \mathbb{C}$ such that for $z \in \mathbb{C} \setminus \mathcal{D}$ and for $f \in L^2(0, L)$, there exists a unique solution $v \in H^3(0, L)$

of the system

$$\begin{cases} v''' + v' + zv = f & \text{in } (0, L), \\ v(0) = v(L) = v'(L) = 0. \end{cases} \tag{2.1}$$

Step 1: An auxiliary shooting problem. For each $z \in \mathbb{C}$, let $U_{(z)} \in C^3(\mathbb{R}; \mathbb{C})$ be the unique solution of the Cauchy problem

$$U_{(z)}''' + U_{(z)}' + zU_{(z)} = 0 \text{ in } (0, L), \quad U_{(z)}'(L) = U_{(z)}(L) = 0, \quad U_{(z)}''(L) = 1. \tag{2.2}$$

Let $\theta: \mathbb{C} \to \mathbb{C}$ be defined by $\theta(z) = U_{(z)}(0)$. Then θ is an entire function. We claim that this function does not vanish identically, and therefore $\mathcal{D} := \theta^{-1}(0)$ is a discrete set. Indeed, let us assume that $U_{(1)}(0) = \theta(1) = 0$. Multiplying (2.2) for $z = 1$ (the equation of $U_{(1)}$) by the (real) function $U_{(1)}$ and integrating by parts on $[0, L]$, one gets

$$\frac{1}{2}U_{(1)}'(0)^2 + \int_0^L U_{(1)}(x)^2 \, dx = 0, \tag{2.3}$$

which implies $U_{(1)} = 0$ in $[0, L]$. This contradicts $U_{(1)}''(L) = 1$.

Step 2: Uniqueness. Let $z \notin \mathcal{D}$, i.e., $\theta(z) = U_{(z)}(0) \neq 0$. Assume that $v_1, v_2 \in H^3(0, L)$ are two solutions of (2.1). Set $U = v_1 - v_2$. Then $U''' + U' + zU = 0$ and $U(L) = U'(L) = 0$. It follows that $U = U''(L)U_{(z)}$ in $[0, L]$. So $U(0) = U''(L)U_{(z)}(0) = U''(L)\theta(z)$. Since $\theta(z) \neq 0$ and $U(0) = v_1(0) - v_2(0) = 0$, we conclude that $U''(L) = 0$. Hence $U = 0$ in $[0, L]$, which implies uniqueness.

Step 3: Existence. Let $z \notin \mathcal{D}$ and $f \in L^2(0, L)$. Let $V \in H^3(0, L)$ be the unique solution of the Cauchy problem

$$\begin{cases} V''' + V' + zV = f & \text{in } (0, L), \\ V(L) = V'(L) = V''(L) = 0. \end{cases} \tag{2.4}$$

Set $v = V - V(0)(\theta(z))^{-1}U_{(z)}$ in $[0, L]$. Then v belongs to $H^3(0, L)$ and satisfies the differential equation $v''' + v' + zv = f$, and the boundary conditions $v(L) = 0$, $v'(L) = 0$, and $v(0) = V(0) - V(0) = 0$. Thus v is a solution of (2.1). ■

Before characterizing controls steering 0 at time 0 to 0 at time T, we introduce

Definition 2.2. For $z \in \mathbb{C}$, let $(\lambda_j)_{1 \le j \le 3} = (\lambda_j(z))_{1 \le j \le 3}$ be the three solutions, repeated with multiplicity, of

$$\lambda^3 + \lambda + iz = 0. \tag{2.5}$$

Set

$$Q = Q(z) := \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \end{pmatrix}, \tag{2.6}$$

$$P = P(z) := \sum_{j=1}^{3} \lambda_j (e^{\lambda_{j+2} L} - e^{\lambda_{j+1} L}) = \det \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}, \tag{2.7}$$

$$\Xi = \Xi(z) := -(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3) = \det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}, \tag{2.8}$$

with the convention $\lambda_{j+3} = \lambda_j$ for $j \geq 1$.

Remark 2.3. The matrix Q and the quantities P and Ξ are antisymmetric with respect to λ_j ($j = 1, 2, 3$), and their definitions depend on the order of $(\lambda_1, \lambda_2, \lambda_3)$. Nevertheless, we later consider a product of either P, Ξ, or $\det Q$ with another antisymmetric function of (λ_j), or deal with $|\det Q|$, and therefore these quantities make sense (see e.g. (2.11), (2.12)). The definitions of P, Ξ, and Q are only understood in these contexts.

In what follows, for an appropriate function v defined on $\mathbb{R}_+ \times (0, L)$, we extend v by 0 on $\mathbb{R}_- \times (0, L)$ and we denote by $\hat{v}$ its Fourier transform with respect to t, i.e., for $z \in \mathbb{C}$,

$$\hat{v}(z, x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} v(t, x) e^{-izt} \, dt.$$

Lemma 2.4. *Let $u \in L^2(0, +\infty)$ and let*

$$y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\text{loc}}([0, +\infty); H^1(0, L))$$

be the unique solution of

$$\begin{cases} y_t(t, x) + y_x(t, x) + y_{xxx}(t, x) = 0 & \text{in } (0, +\infty) \times (0, L), \\ y(t, x = 0) = y(t, x = L) = 0 & \text{in } (0, +\infty), \\ y_x(t, x = L) = u(t) & \text{in } (0, +\infty), \end{cases} \tag{2.9}$$

with

$$y(t = 0, \cdot) = 0 \quad \text{in } (0, L). \tag{2.10}$$

Then, outside of a discrete set $z \in \mathbb{R}$, we have

$$\hat{y}(z, x) = \frac{\hat{u}}{\det Q} \sum_{j=1}^{3} (e^{\lambda_{j+2} L} - e^{\lambda_{j+1} L}) e^{\lambda_j x} \quad \text{for a.e. } x \in (0, L), \tag{2.11}$$

and in particular

$$\partial_x \hat{y}(z, 0) = \frac{\hat{u}(z) P(z)}{\det Q(z)}. \tag{2.12}$$

Remark 2.5. Assume that $\hat{u}(z, \cdot)$ is well-defined for $z \in \mathbb{C}$ (e.g. when u has a compact support). Then the conclusions of Lemma 2.4 hold outside of a discrete set of $z \in \mathbb{C}$.

Proof of Lemma 2.4. From the system of y, we have

$$\begin{cases} iz\hat{y}(z,x) + \hat{y}_x(z,x) + \hat{y}_{xxx}(z,x) = 0 & \text{in } \mathbb{R} \times (0,L), \\ \hat{y}(z,x=0) = \hat{y}(z,x=L) = 0 & \text{in } \mathbb{R}, \\ \hat{y}_x(z,x=L) = \hat{u}(z) & \text{in } \mathbb{R}. \end{cases} \tag{2.13}$$

Taking into account the equation of $\hat{y}$, we search for the solution of the form

$$\hat{y}(z,\cdot) = \sum_{j=1}^{3} a_j e^{\lambda_j x},$$

where $\lambda_j = \lambda_j(z)$ with $j = 1,2,3$ are defined in Definition 2.2.

According to the theory of ordinary differential equations with constant coefficients, this is possible if the equation $\lambda^3 + \lambda + iz = 0$ has three distinct solutions, i.e., if the discriminant $-4 + 27z^2$ is not 0. Moreover, if $-iz \notin \mathrm{Sp}(\mathcal{A})$, this solution is unique. Thus, by Lemma 2.1, outside a discrete set in $\mathbb{R}$, $\hat{y}(z,\cdot)$ can be written in this form in a unique way. Using the boundary conditions for $\hat{y}$, we require that

$$\begin{cases} \sum_{j=1}^{3} a_j = 0, \\ \sum_{j=1}^{3} e^{\lambda_j L} a_j = 0, \\ \sum_{j=1}^{3} \lambda_j e^{\lambda_j L} a_j = \hat{u}. \end{cases}$$

This implies, with $Q = Q(z)$ defined in Definition 2.2,

$$Q(a_1,a_2,a_3)^{\mathsf{T}} = (0,0,\hat{u})^{\mathsf{T}}. \tag{2.14}$$

It follows that

$$a_j = \frac{\hat{u}}{\det Q}(e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}).$$

This yields

$$\hat{y}(z,x) = \frac{\hat{u}}{\det Q} \sum_{j=1}^{3} (e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}) e^{\lambda_j x}. \tag{2.15}$$

We thus obtain

$$\partial_x \hat{y}(z,0) = \frac{\hat{u}(z)P(z)}{\det Q(z)}. \tag{2.16}$$

■

As mentioned in Remark 2.3, the maps P and $\det Q$ are antisymmetric functions with respect to λ_j. It is hence convenient to write $\partial_x \hat{y}(z,0)$ in the form

$$\partial_x \hat{y}(z,0) = \frac{\hat{u}(z)G(z)}{H(z)}, \tag{2.17}$$

where, with Ξ defined in (2.8),

$$G(z) = P(z)/\Xi(z) \quad \text{and} \quad H(z) = \det Q(z)/\Xi(z). \tag{2.18}$$

Concerning the functions G and H, we have

Lemma 2.6. *The functions G and H defined in (2.18) are entire functions.*

Proof. Note that the maps $z \mapsto \Xi(z)P(z)$, $z \mapsto \Xi(z)\det Q(z)$ and $z \mapsto \Xi(z)^2$ are symmetric functions of the λ_j and are thus well-defined, and even entire functions (see Lemma A.1 in Appendix A). According to the definition of Ξ, $\Xi(z_0) = 0$ if and only if $X^3 + X + iz_0$ has a double root, i.e. $z_0 = \pm 2/(3\sqrt{3})$. Simple computations prove that when ϵ is small,

$$\begin{cases} \lambda_1(z_0+\varepsilon) = \mp \dfrac{i}{\sqrt{3}} + \dfrac{\sqrt{\mp i}}{3^{1/4}}\sqrt{\epsilon} + O(\varepsilon), \\ \lambda_2(z_0+\varepsilon) = \mp \dfrac{i}{\sqrt{3}} - \dfrac{\sqrt{\mp i}}{3^{1/4}}\sqrt{\varepsilon} + O(\varepsilon), \\ \lambda_3(z_0+\varepsilon) = \pm \dfrac{2i}{\sqrt{3}} + \dfrac{\varepsilon}{3} + O(\varepsilon^2). \end{cases} \tag{2.19}$$

Indeed, the behavior of λ_3 follows immediately from the expansion of λ_3 near $\pm\frac{2i}{\sqrt{3}}$. The behavior of λ_1 and λ_2 can then be verified using, with $\Delta = -3\lambda_3^2 - 4$,

$$\lambda_1 = \frac{-\lambda_3 + \sqrt{\Delta}}{2} \quad \text{and} \quad \lambda_2 = \frac{-\lambda_3 - \sqrt{\Delta}}{2}.$$

It follows that that $\Xi(z_0+\varepsilon)^2 = c_\pm \varepsilon + O(\varepsilon^2)$ for some $c_\pm \neq 0$. This in turn implies that $z_0 = \pm 2/(3\sqrt{3})$ are simple zeros of Ξ^2. When $X^3 + X + iz$ has a double root, the definitions of P and $\det Q$ (in (2.6) and (2.7)) imply

$$|P(z_0)| = |\det Q(z_0)| = 0 \quad \text{for } z_0 = \pm 2/(3\sqrt{3}).$$

The conclusion follows. ■

Remark 2.7. It is interesting to note that

(1) ($H(z) = 0$ and $z \neq \pm 2/(3\sqrt{3})$) if and only if $-iz \in \mathrm{Sp}(\mathcal{A})$;

(2) $iz \in \mathrm{Sp}(\mathcal{A})$ and z is real if and only if $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$, and

$$z = \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}\,(k^2+kl+l^2)^{3/2}}, \tag{2.20}$$

for some $k, l \in \mathbb{N}_*$ with $1 \leq l \leq k$.

Indeed, if $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and z is given by the RHS of (2.20), then, from [38], $iz \in \mathrm{Sp}(\mathcal{A})$. On the other hand, if z is real and $iz \in \mathrm{Sp}(\mathcal{A})$, then, by an integration by parts, the corresponding eigenfunction w also satisfies the condition $w_x(0) = 0$. It follows from [38] that $L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and z is given by (2.20) for some $k, l \in \mathbb{N}_*$ with $1 \leq l \leq k$. We finally note that for $z \neq \pm 2/(3\sqrt{3})$, the solutions of the ordinary differential equation $u''' + u' + izu = 0$ are of the form $u(x) = \sum_{j=1}^3 a_j e^{\lambda_j x}$. This implies that $Q(a_1, a_2, a_3)^\mathsf{T} = (0,0,0)^\mathsf{T}$ if $u(0) = u(L) = u'(L) = 0$. Therefore, for $z \neq \pm 2/(3\sqrt{3})$, $-iz$ is an eigenvalue of $\mathcal{A}$ if and only if $|\det Q(z)| = 0$, i.e., $H(z) = 0$. We finally note that $\pm 2i/(3\sqrt{3})$ is not a purely imaginary eigenvalue of $\mathcal{A}$ since, for $k \geq l \geq 1$,

$$0 \leq \frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}\,(k^2+kl+l^2)^{3/2}} = \frac{(2k+l)(k^2+kl-2l^2)}{3\sqrt{3}\,(k^2+kl+l^2)^{3/2}} < \frac{(2k+l)}{3\sqrt{3}\,(k^2+kl+l^2)^{1/2}} < \frac{2}{3\sqrt{3}}.$$

We are ready to give the characterization of the controls steering 0 to 0, which is the starting point of our analysis.

Proposition 2.8. *Let $L, T > 0$, and $u \in L^2(0,+\infty)$. Assume that u has a compact support in $[0,T]$, and u steers 0 at time 0 to 0 at time T, i.e., the unique solution y of* (2.9)–(2.10) *satisfies $y(T,\cdot) = 0$ in $(0,L)$. Then $\hat{u}$ and $\hat{u}G/H$ satisfy the assumptions of Paley–Wiener's theorem concerning the support in $[-T,T]$, i.e.,*

$$\hat{u} \text{ and } \hat{u}G/H \text{ are entire functions},$$

and

$$|\hat{u}(z)| + \left|\frac{\hat{u}G(z)}{H(z)}\right| \leq Ce^{T|\Im(z)|}$$

for some positive constant C.

Here and in what follows, for a complex number z, $\Re(z)$, $\Im(z)$, and $\bar{z}$ denote the real part, the imaginary part, and the conjugate of z, respectively.

Proof of Proposition 2.8. Proposition 2.8 is a consequence of Lemma 2.4 and Paley–Wiener's theorem (see e.g. [41, 19.3 Theorem]). The proof is clear from the analysis above in this section and left to the reader. ∎

3. Attainable directions for small time

In this section, we investigate controls which steer a linear KdV equation from 0 to 0 in some time T, and a quantity related to the quadratic order in the power expansion of a nonlinear KdV equation. Let $u \in L^2(0,+\infty)$ and denote by y the corresponding solution of the linear KdV equation (2.9). We assume that the initial condition is 0 and y satisfies $y(t,\cdot) = 0$ in $(0,L)$ for $t \geq T$. We have, by Lemma 2.4 (and also Remark 2.5), for $z \in \mathbb{C}$ outside a discrete set,

$$\hat{y}(z,x) = \hat{u}(z)\frac{\sum_{j=1}^{3}(e^{\lambda_{j+1}L} - e^{\lambda_j L})e^{\lambda_{j+2}x}}{\sum_{j=1}^{3}(\lambda_{j+1} - \lambda_j)e^{-\lambda_{j+2}L}}. \tag{3.1}$$

Recall that $\lambda_j = \lambda_j(z)$ for $j = 1, 2, 3$ are the three solutions of the equation

$$x^3 + x = -iz \quad \text{for } z \in \mathbb{C}. \tag{3.2}$$

Let $\eta_1, \eta_2, \eta_3 \in i\mathbb{R}$, i.e., $\eta_j \in \mathbb{C}$ with $\Re(\eta_j) = 0$ for $j = 1,2,3$. Define

$$\varphi(x) = \sum_{j=1}^{3}(\eta_{j+1} - \eta_j)e^{\eta_{j+2}x} \quad \text{for } x \in [0,L], \tag{3.3}$$

with the convention $\eta_{j+3} = \eta_j$ for $j \geq 1$. The following assumption on η_j is used repeatedly throughout the paper:

$$e^{\eta_1 L} = e^{\eta_2 L} = e^{\eta_3 L}, \tag{3.4}$$

which is equivalent to $\eta_3 - \eta_2, \eta_2 - \eta_1 \in \frac{2\pi i}{L}\mathbb{Z}$. The definition of φ in (3.3) and the assumption on η_j in (3.4) are motivated by the structure of $\mathcal{M}$ [18, 20] and will be clear in Section 5.

We have

Lemma 3.1. *Let $p \in \mathbb{R}$ and let φ be defined by (3.3). Set, for $(z, x) \in \mathbb{C} \times [0, L]$,*

$$B(z,x) = \frac{\sum_{j=1}^{3}(e^{\lambda_{j+1}L} - e^{\lambda_j L})e^{\lambda_{j+2}x}}{\sum_{j=1}^{3}(\lambda_{j+1} - \lambda_j)e^{-\lambda_{j+2}L}} \cdot \frac{\sum_{j=1}^{3}(e^{\widetilde{\lambda}_{j+1}L} - e^{\widetilde{\lambda}_j L})e^{\widetilde{\lambda}_{j+2}x}}{\sum_{j=1}^{3}(\widetilde{\lambda}_{j+1} - \widetilde{\lambda}_j)e^{-\widetilde{\lambda}_{j+2}L}} \cdot \varphi_x(x), \tag{3.5}$$

where $\widetilde{\lambda}_j = \widetilde{\lambda}_j(z)$ ($j = 1, 2, 3$) denotes the conjugate of the roots of (3.2) with z replaced by $z - p$ and with the use of the convention $\widetilde{\lambda}_{j+3} = \widetilde{\lambda}_j$ for $j \geq 1$. Let $u \in L^2(0, +\infty)$ and let $y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\mathrm{loc}}([0, +\infty); H^1(0, L))$ be the unique solution of (2.9)–(2.10). Assume that $u(t) = 0$ and $y(t, \cdot) = 0$ for large t. Then

$$\int_0^L \int_0^{+\infty} |y(t,x)|^2 \varphi_x(x) e^{-ipt}\,dt\,dx = \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz. \tag{3.6}$$

Remark 3.2. The LHS of (3.6) is a multiple of the $L^2(0, L)$-projection of the solution $y(T, \cdot)$ into the space spanned by the conjugate of the vector $\varphi(x)e^{-ipT}$ whose real and imaginary parts are in $\mathcal{M}$ for appropriate choices of η_j and p when the initial data is orthogonal to $\mathcal{M}$ (see [18, 20, 24], and also (5.18)).

Proof of Lemma 3.1. We have

$$\begin{aligned}
&\int_0^L \int_0^{\infty} |y(t,x)|^2 \varphi_x(x) e^{-ipt}\,dt\,dx \\
&= \sqrt{2\pi} \int_0^L \varphi_x(x) \widehat{|y|^2}(p,x)\,dx = \int_0^L \varphi_x(x) \hat{y} * \hat{\bar{y}}(p,x)\,dx \\
&= \int_0^L \varphi_x(x) \int_{\mathbb{R}} \hat{y}(z,x)\hat{\bar{y}}(p-z,x)\,dz\,dx = \int_0^L \varphi_x(x) \int_{\mathbb{R}} \hat{y}(z,x)\overline{\hat{y}}(z-p,x)\,dz\,dx.
\end{aligned}$$

Using Fubini's theorem, we deduce from (3.1) that

$$\int_0^L \int_0^{\infty} |y(t,x)|^2 \varphi_x(x) e^{-ipt}\,dt\,dx = \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz,$$

which is (3.6). ∎

We next state the behaviors of λ_j and $\widetilde{\lambda}_j$ given in Lemma 3.1 for "large positive" z, which will be used repeatedly in this section and in Section 4. These asymptotics are direct consequences of equation (2.5) satisfied by the λ_j.

Lemma 3.3. *For $p \in \mathbb{R}$ and z in a small enough conic neighborhood of $\mathbb{R}_+$, let λ_j and $\widetilde{\lambda}_j$ with $j = 1, 2, 3$ be given in Lemma 3.1. Consider the convention $\Re(\lambda_1) < \Re(\lambda_2) < \Re(\lambda_3)$ and similarly for $\widetilde{\lambda}_j$. We have, in the limit $|z| \to \infty$,*

$$\lambda_j = \mu_j z^{1/3} - \frac{1}{3\mu_j} z^{-1/3} + O(z^{-2/3}) \quad \textit{with } \mu_j = e^{-i\pi/6 - 2ji\pi/3}, \tag{3.7}$$

$$\widetilde{\lambda}_j = \widetilde{\mu}_j z^{1/3} - \frac{1}{3\widetilde{\mu}_j} z^{-1/3} + O(z^{-2/3}) \quad \textit{with } \widetilde{\mu}_j = e^{i\pi/6 + 2ij\pi/3} \tag{3.8}$$

(see Figure 1 for the geometry of μ_j and $\widetilde{\mu}_j$). Here $z^{1/3}$ denotes the cube root of z with the real part positive.

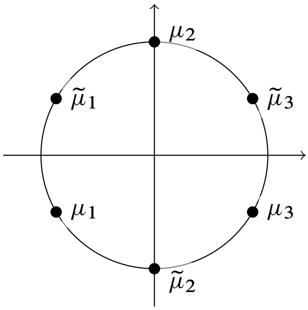

Fig. 1. The roots λ_j of $\lambda^3 + \lambda + iz = 0$ satisfy, when $z > 0$ is large, $\lambda_j \sim \mu_j z^{1/3}$ where $\mu_j^3 = -i$. When $z < 0$ and $|z|$ is large, the corresponding roots $\hat{\lambda}_j$ satisfy $\hat{\lambda}_j \sim \widetilde{\mu}_j |z|^{1/3}$ with $\widetilde{\mu}_j = \bar{\mu}_j$. We also have $\widetilde{\lambda}_j \sim \hat{\lambda}_j$.

We are ready to establish the behavior of

$$\int_0^L B(z, x)\, dx$$

for $z \in \mathbb{R}$ with large $|z|$, which is one of the main ingredients for the analysis in this section.

Lemma 3.4. *Let $p \in \mathbb{R}$, and let φ be defined by (3.3). Assume that (3.4) holds and $\eta_j \neq 0$ for $j = 1, 2, 3$. Let B be defined by (3.5). We have*

$$\int_0^L B(z, x)\, dx = \frac{E}{|z|^{4/3}} + O(|z|^{-5/3}) \quad \textit{for } z \in \mathbb{R} \textit{ with large } |z|, \tag{3.9}$$

where E is defined by

$$E = \frac{1}{3}(e^{\eta_1 L} - 1)\left(-\frac{2}{3}\sum_{j=1}^{3} \eta_{j+2}^2 (\eta_{j+1} - \eta_j) - ip \sum_{j=1}^{3} \frac{\eta_{j+1} - \eta_j}{\eta_{j+2}}\right). \tag{3.10}$$

Proof. We first deal with the case where z is positive and large. We use the convention in Lemma 3.3 for λ_j and $\widetilde{\lambda}_j$. Consider the denominator of $B(z, x)$. We have, by Lemma 3.3,

$$\frac{1}{\sum_{j=1}^3 (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2} L}} \cdot \frac{1}{\sum_{j=1}^3 (\widetilde{\lambda}_{j+1} - \widetilde{\lambda}_j) e^{-\widetilde{\lambda}_{j+2} L}} = \frac{e^{\lambda_1 L} e^{\widetilde{\lambda}_1 L}}{(\lambda_3 - \lambda_2)(\widetilde{\lambda}_3 - \widetilde{\lambda}_2)} (1 + O(e^{-C|z|^{1/3}})). \tag{3.11}$$

We next deal with the numerator of $B(z,x)$. Set,[3] for $(z,x) \in \mathbb{R} \times (0,L)$,

$$f(z,x) = \sum_{j=1}^{3} (e^{\lambda_{j+1}L} - e^{\lambda_j L}) e^{\lambda_{j+2}x}, \quad g(z,x) = \sum_{j=1}^{3} (e^{\widetilde{\lambda}_{j+1}L} - e^{\widetilde{\lambda}_j L}) e^{\widetilde{\lambda}_{j+2}x}, \tag{3.12}$$

$$f_m(z,x) = -e^{\lambda_3 L} e^{\lambda_2 x} + e^{\lambda_2 L} e^{\lambda_3 x} + e^{\lambda_3 L} e^{\lambda_1 x},$$
$$g_m(z,x) = -e^{\widetilde{\lambda}_3 L} e^{\widetilde{\lambda}_2 x} + e^{\widetilde{\lambda}_2 L} e^{\widetilde{\lambda}_3 x} + e^{\widetilde{\lambda}_3 L} e^{\widetilde{\lambda}_1 x}.$$

We have

$$\begin{aligned}
&\int_0^L f(z,x) g(z,x) \varphi_x(x)\, dx \\
&= \int_0^L f_m(z,x) g_m(z,x) \varphi_x(x)\, dx + \int_0^L (f - f_m)(z,x) g_m(z,x) \varphi_x(x)\, dx \\
&\quad + \int_0^L f_m(z,x)(g - g_m)(z,x) \varphi_x(x)\, dx + \int_0^L (f - f_m)(z,x)(g - g_m)(z,x) \varphi_x(x)\, dx.
\end{aligned}$$

It is clear from Lemma 3.3 that

$$\begin{aligned}
&\int_0^L |(f - f_m)(z,x) g_m(z,x) \varphi_x(x)|\, dx + \int_0^L |(f - f_m)(z,x)(g - g_m)(z,x) \varphi_x(x)|\, dx \\
&\quad + \int_0^L |f_m(z,x)(g - g_m)(z,x) \varphi_x(x)|\, dx \le C |e^{(\lambda_3 + \widetilde{\lambda}_3)L}| e^{-C|z|^{1/3}}.
\end{aligned} \tag{3.13}$$

We next estimate

$$\int_0^L f_m(x,z) g_m(x,z) \varphi_x(x) = \int_0^L f_m(x,z) g_m(x,z) \Big(\sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) e^{\eta_{j+2}x} \Big)\, dx. \tag{3.14}$$

We first have, by (3.4) and Lemma 3.3,

$$\begin{aligned}
&\int_0^L (-e^{\lambda_3 L} e^{\lambda_2 x} e^{\widetilde{\lambda}_2 L} e^{\widetilde{\lambda}_3 x} - e^{\lambda_2 L} e^{\lambda_3 x} e^{\widetilde{\lambda}_3 L} e^{\widetilde{\lambda}_2 x} + e^{\lambda_2 L} e^{\lambda_3 x} e^{\widetilde{\lambda}_2 L} e^{\widetilde{\lambda}_3 x}) \\
&\times \Big(\sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) e^{\eta_{j+2}x} \Big)\, dx = e^{(\lambda_3 + \widetilde{\lambda}_3 + \lambda_2 + \widetilde{\lambda}_2)L} \big(e^{\eta_1 L} T_1(z) + O(e^{-C|z|^{1/3}}) \big),
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
&T_1(z) \\
&:= \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) \left(\frac{1}{\lambda_3 + \widetilde{\lambda}_3 + \eta_{j+2}} - \frac{1}{\lambda_3 + \widetilde{\lambda}_2 + \eta_{j+2}} - \frac{1}{\lambda_2 + \widetilde{\lambda}_3 + \eta_{j+2}} \right).
\end{aligned} \tag{3.16}$$

[3] The index m stands for the main part.

Let us now deal with the terms of (3.14) that contain both $e^{\lambda_3 L+\widetilde{\lambda}_3 L}$ and either $e^{\lambda_1 x}$ or $e^{\widetilde{\lambda}_1 x}$. We obtain, by (3.4) and Lemma 3.3,

$$\int_0^L (e^{\lambda_3 L}e^{\lambda_1 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_1 x} - e^{\lambda_3 L}e^{\lambda_1 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_2 x} - e^{\lambda_3 L}e^{\lambda_2 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_1 x})$$

$$\times \Big(\sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1}-\eta_j)e^{\eta_{j+2}x}\Big)\,dx = e^{(\lambda_3+\widetilde{\lambda}_3)L}(T_2(z) + O(e^{-C|z|^{1/3}})), \tag{3.17}$$

where

$$T_2(z)$$

$$:= \sum_{j=1}^{3}\eta_{j+2}(\eta_{j+1}-\eta_j)\Big(-\frac{1}{\lambda_1+\widetilde{\lambda}_1+\eta_{j+2}} + \frac{1}{\lambda_1+\widetilde{\lambda}_2+\eta_{j+2}} + \frac{1}{\lambda_2+\widetilde{\lambda}_1+\eta_{j+2}}\Big). \tag{3.18}$$

We have, by (3.4),

$$\int_0^L e^{\lambda_3 L}e^{\lambda_2 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_2 x}\Big(\sum_{j=1}^{3}\eta_{j+2}(\eta_{j+1}-\eta_j)e^{\eta_{j+2}x}\Big)\,dx = e^{(\lambda_3+\widetilde{\lambda}_3)L}T_3(z), \tag{3.19}$$

where

$$T_3(z) := (e^{\lambda_2 L+\widetilde{\lambda}_2 L+\eta_1 L}-1)\sum_{j=1}^{3}\frac{\eta_{j+2}(\eta_{j+1}-\eta_j)}{\lambda_2+\widetilde{\lambda}_2+\eta_{j+2}}. \tag{3.20}$$

The other terms of (3.14) are negligible, because we have

$$\Big|\int_0^L (e^{\lambda_3 L}e^{\lambda_1 x}e^{\widetilde{\lambda}_2 L}e^{\widetilde{\lambda}_3 x} + e^{\lambda_2 L}e^{\lambda_3 x}e^{\widetilde{\lambda}_3 L}e^{\widetilde{\lambda}_1 x})\Big(\sum_{j=1}^{3}\eta_{j+2}(\eta_{j+1}-\eta_j)e^{\eta_{j+2}x}\Big)\,dx\Big|$$

$$= |e^{(\lambda_3+\widetilde{\lambda}_3)L}|\,O(e^{-Cz^{1/3}}). \tag{3.21}$$

Using Lemma 3.3, we have

$$\begin{cases} \lambda_1+\widetilde{\lambda}_1+\lambda_2+\widetilde{\lambda}_2+\lambda_3+\widetilde{\lambda}_3 = O(z^{-1/3}), \\ \lambda_1+\widetilde{\lambda}_1+\lambda_3+\widetilde{\lambda}_3 = O(z^{-1/3}), \\ (\lambda_3-\lambda_2)(\widetilde{\lambda}_3-\widetilde{\lambda}_2) = 3z^{2/3}(1+O(z^{-1/3})). \end{cases} \tag{3.22}$$

We claim that

$$|T_1(z)| + |T_2(z)| + |T_3(z)| = O(z^{-2/3}) \quad \text{for large positive } z. \tag{3.23}$$

Assuming (3.23), and combining (3.11), (3.15), (3.17), (3.19), (3.21), and (3.22) yields

$$\int_0^L B(z,x)\,dz = \frac{1}{3|z|^{2/3}}\big(e^{\eta_1 L}T_1(z) + T_2(z) + T_3(z) + O(z^{-1})\big). \tag{3.24}$$

We next derive the asymptotic behaviors of $T_1(z)$, $T_2(z)$, and $T_3(z)$, which in particular imply (3.23). We first deal with $T_1(z)$ given in (3.16). Since

$$\sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j) = 0, \tag{3.25}$$

we obtain

$$\begin{aligned}
T_1(z) = &\sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j)\left(\frac{1}{\lambda_3 + \widetilde{\lambda}_3 + \eta_{j+2}} - \frac{1}{\lambda_3 + \widetilde{\lambda}_3}\right)\\
&+ \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j)\left(-\frac{1}{\lambda_3 + \widetilde{\lambda}_2 + \eta_{j+2}} + \frac{1}{\lambda_3 + \widetilde{\lambda}_2}\right)\\
&+ \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j)\left(-\frac{1}{\lambda_2 + \widetilde{\lambda}_3 + \eta_{j+2}} + \frac{1}{\lambda_2 + \widetilde{\lambda}_3}\right).
\end{aligned}$$

Using Lemma 3.3, we get

$$T_1(z) = -\sum_{j=1}^{3} \eta_{j+2}^2(\eta_{j+1} - \eta_j)\left(\frac{1}{(\lambda_3 + \widetilde{\lambda}_3)^2} - \frac{1}{(\lambda_3 + \widetilde{\lambda}_2)^2} - \frac{1}{(\lambda_2 + \widetilde{\lambda}_3)^2}\right) + O(z^{-1}).$$

Moreover, from Lemma 3.3 we derive

$$\begin{aligned}
&\frac{1}{(\lambda_3 + \widetilde{\lambda}_3)^2} - \frac{1}{(\lambda_3 + \widetilde{\lambda}_2)^2} - \frac{1}{(\lambda_2 + \widetilde{\lambda}_3)^2}\\
&\quad = z^{-2/3}\big((\mu_3 + \widetilde{\mu}_3)^{-2} - (\mu_3 + \widetilde{\mu}_2)^{-2} - (\mu_2 + \widetilde{\mu}_3)^{-2}\big) + O(z^{-1})\\
&\quad = z^{-2/3}\left(\frac{1}{3} - \frac{-1 + i\sqrt{3}}{6} - \frac{-1 - i\sqrt{3}}{6}\right) + O(z^{-1}) = \frac{2}{3} z^{-2/3} + O(z^{-1}).
\end{aligned} \tag{3.26}$$

We deduce that

$$T_1(z) = -\frac{2}{3} z^{-2/3} \sum_{j=1}^{3} \eta_{j+2}^2(\eta_{j+1} - \eta_j) + O(z^{-1}). \tag{3.27}$$

We next consider $T_2(z)$ given in (3.18). We have, by (3.25),

$$\begin{aligned}
T_2(z) = &\sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j)\left(-\frac{1}{\lambda_1 + \widetilde{\lambda}_1 + \eta_{j+2}} + \frac{1}{\lambda_1 + \widetilde{\lambda}_1}\right)\\
&+ \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j)\left(\frac{1}{\lambda_1 + \widetilde{\lambda}_2 + \eta_{j+2}} - \frac{1}{\lambda_1 + \widetilde{\lambda}_2}\right)\\
&+ \sum_{j=1}^{3} \eta_{j+2}(\eta_{j+1} - \eta_j)\left(\frac{1}{\lambda_2 + \widetilde{\lambda}_1 + \eta_{j+2}} - \frac{1}{\lambda_2 + \widetilde{\lambda}_1}\right).
\end{aligned}$$

Using Lemma 3.3, we obtain

$$T_2(z) = \sum_{j=1}^{3} \eta_{j+2}^2(\eta_{j+1} - \eta_j)\left(\frac{1}{(\lambda_1 + \widetilde{\lambda}_1)^2} - \frac{1}{(\lambda_1 + \widetilde{\lambda}_2)^2} - \frac{1}{(\lambda_2 + \widetilde{\lambda}_1)^2}\right) + O(z^{-1}),$$

and

$$\frac{1}{(\lambda_1 + \widetilde{\lambda}_1)^2} - \frac{1}{(\lambda_1 + \widetilde{\lambda}_2)^2} - \frac{1}{(\lambda_2 + \widetilde{\lambda}_1)^2}$$
$$= z^{-2/3}\big((\mu_1 + \widetilde{\mu}_1)^{-2} - (\mu_1 + \widetilde{\mu}_2)^{-2} - (\mu_2 + \widetilde{\mu}_1)^{-2}\big) + O(z^{-1}).$$

By Lemma 3.3, we have

$$(\mu_1 + \widetilde{\mu}_1)^2 = (\mu_3 + \widetilde{\mu}_3)^2 \quad (\mu_1 + \widetilde{\mu}_2)^2 = (\widetilde{\mu}_3 + \mu_2)^2 \quad (\widetilde{\mu}_1 + \mu_2)^2 = (\mu_3 + \widetilde{\mu}_2)^2.$$

Combining this with (3.26), we then have

$$T_2(z) = \frac{2}{3} z^{-2/3} \sum_{j=1}^{3} \eta_{j+2}^2(\eta_{j+1} - \eta_j) + O(z^{-1}). \tag{3.28}$$

We finally consider $T_3(z)$ given in (3.20). We have, by (2.5),

$$\lambda_2^3 + \widetilde{\lambda}_2^3 + \lambda_2 + \widetilde{\lambda}_2 = -iz + i(z - p) = -ip.$$

This yields

$$\lambda_2 + \widetilde{\lambda}_2 = -\frac{ip}{\lambda_2^2 + \widetilde{\lambda}_2^2 + \lambda_2\widetilde{\lambda}_2}.$$

From Lemma 3.3, we have

$$\lambda_2 + \widetilde{\lambda}_2 = ipz^{-2/3} + O(z^{-1}).$$

It follows that

$$\sum_{j=1}^{3} \frac{\eta_{j+2}(\eta_{j+1} - \eta_j)}{\lambda_2 + \widetilde{\lambda}_2 + \eta_{j+2}} = \sum_{j=1}^{3} \frac{\eta_{j+2}(\eta_{j+1} - \eta_j)}{ipz^{-2/3} + \eta_{j+2}} + O(|z|^{-1})$$
$$= \sum_{j=1}^{3} (\eta_{j+1} - \eta_j)\left(1 - \frac{ipz^{-2/3}}{\eta_{j+2}}\right) + O(|z|^{-1})$$
$$= -ip \sum_{j=1}^{3} \frac{\eta_{j+1} - \eta_j}{\eta_{j+2}} z^{-2/3} + O(z^{-1}). \tag{3.29}$$

We deduce from (3.29) and Lemma 3.3 that

$$T_3 = -ip(e^{\eta_1 L} - 1) \sum_{j=1}^{3} \frac{\eta_{j+1} - \eta_j}{\eta_{j+2}} z^{-2/3} + O(z^{-1}). \tag{3.30}$$

Using (3.27), (3.28), and (3.30), we infer from (3.24) that

$$\int_0^L B(z, x)\, dx = Ez^{-4/3} + O(z^{-5/3}),$$

which is the conclusion for large positive z.

The conclusion in the case where z is large and negative can be derived from the case where z is positive and large as follows. Define, for $(z,x) \in \mathbb{R} \times (0,L)$ with large $|z|$,

$$M(z,x) = \frac{\sum_{j=1}^{3} (e^{\lambda_{j+1}L} - e^{\lambda_j L}) e^{\lambda_{j+2}x}}{\sum_{j=1}^{3} (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2}L}}.$$

Then

$$B(z,x) = M(z,x)\overline{M(z-p,x)}\varphi_x(x).$$

It is clear from the definition of M that

$$M(-z,x) = \overline{M(z,x)}.$$

We then have

$$B(-z,x) = M(-z,x)\overline{M(-z-p,x)}\varphi_x(x) = \overline{M(z,x)\overline{M(z+p,x)}\,\overline{\varphi_x(x)}}.$$

We thus obtain the result in the case where z is negative and large by taking the conjugate of the corresponding expression for large positive z in which η_j and p are replaced by $-\eta_j$ and $-p$. The conclusion follows. ∎

As a consequence of Lemmas 3.1 and 3.4, we obtain

Lemma 3.5. *Let $p \in \mathbb{R}$ and let φ be defined by (3.3). Assume that (3.4) holds and $\eta_j \neq 0$ for $j = 1,2,3$. Let $u \in L^2(0,+\infty)$ and let $y \in C([0,+\infty); L^2(0,L)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^1(0,L))$ be the unique solution of (2.9)–(2.10). Assume that $u(t) = 0$ and $y(t,\cdot) = 0$ for large t. We have*

$$\int_0^{+\infty} \int_0^L |y(t,x)|^2 \varphi_x(x) e^{-ipt}\,dx\,dt = \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)}\left(\frac{E}{|z|^{4/3}} + O(|z|^{-5/3})\right) dz. \tag{3.31}$$

Using Lemma 3.5, we will establish the following result which is the key ingredient in the analysis of the non-null-controllability for small time of the KdV system (1.1).

Proposition 3.6. *Let $p \in \mathbb{R}$ and let φ be defined by (3.3). Assume that (3.4) holds and $\eta_j \neq 0$ for $j = 1,2,3$. Let $u \in L^2(0,+\infty)$ and let $y \in C([0,+\infty); L^2(0,L)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^1(0,L))$ be the unique solution of (2.9)–(2.10). Assume that $u \not\equiv 0$, $u(t) = 0$ for $t > T$, and $y(t,\cdot) = 0$ for large t. Then there exists a real number $N(u) \geq 0$ such that $C^{-1}\|u\|_{H^{-2/3}} \leq N(u) \leq C\|u\|_{H^{-2/3}}$ for some constant $C \geq 1$ depending only on L, and*[4]

$$\int_0^{\infty} \int_0^L |y(t,x)|^2 e^{-ipt} \varphi_x(x)\,dx\,dt = N(u)^2 (E + O(1)T^{1/4}). \tag{3.32}$$

[4]The map $u \mapsto N(u)$ is actually a norm, which is (somewhat indirectly) given in the proof, by $N(u)^2 = \|\hat{w}\|^2_{L^2}$, where w is defined in (3.46).

Here we use the following definition, for $s < 0$ and for $u \in L^2(\mathbb{R}_+)$:

$$\|u\|^2_{H^s(\mathbb{R})} = \int_{\mathbb{R}} |\hat{u}|^2 (1 + |\xi|^2)^s \, d\xi,$$

where $\hat{u}$ is the Fourier transform of the extension of u by 0 for $t < 0$.

Before giving the proof of Proposition 3.6, we present one of its direct consequences. Denote $\xi_1(t, x) = \Re\{\varphi(x) e^{-ipt}\}$ and $\xi_2(t, x) = \Im\{\varphi(x) e^{-ipt}\}$. Then

$$\xi_1(t, x) + i\xi_2(t, x) = \varphi(x) e^{-ipt}. \tag{3.33}$$

Denote $E_1 = \Re(E)$ and $E_2 = \Im(E)$, and set

$$\Psi(t, x) = E_1 \xi_1(t, x) + E_2 \xi_2(t, x). \tag{3.34}$$

Multiplying (3.32) by $\overline{E}$ and normalizing appropriately, we have

Corollary 3.7. *Let $p \in \mathbb{R}$ and let φ be defined by (3.3). Assume that (3.4) holds, $\eta_j \neq 0$ for $j = 1, 2, 3$, and $E \neq 0$. There exists $T_* > 0$ such that, for any (real) $u \in L^2(0, +\infty)$ with $u(t) = 0$ for $t > T_*$ and $y(t, \cdot) = 0$ for large t where y is the unique solution of (2.9)–(2.10), we have*

$$\int_0^\infty \int_0^{+\infty} y(t, x)^2 \Psi_x(t, x) \, dx \, dt \geq C \|u\|^2_{H^{-2/3}(\mathbb{R})}. \tag{3.35}$$

Proof of Proposition 3.6. By Proposition 2.8,

$$\hat{u} G / H \text{ is an entire function.}$$

By Lemma 2.6, G and H are entire functions. The same holds for $\hat{u}$ since $u(t) = 0$ for large t. One can show that the number of common roots of G and H in $\mathbb{C}$ is finite, see Lemma B.2. Let $z_1, \dots, z_k$ be the distinct common roots of G and H in $\mathbb{C}$. There exist $m_1, \dots, m_k \in \mathbb{N}_*$ such that,[5] with

$$\Gamma(z) = \prod_{j=1}^{k} (z - z_j)^{m_j} \quad \text{in } \mathbb{C},$$

the following two functions are entire:

$$\mathcal{G}(z) := \frac{G(z)}{\Gamma(z)} \quad \text{and} \quad \mathcal{H}(z) := \frac{H(z)}{\Gamma(z)}, \tag{3.36}$$

and $\mathcal{G}$ and $\mathcal{H}$ have no common roots. Since

$$\hat{u} \mathcal{G} / \mathcal{H} = \hat{u} G / H$$

[5] One can prove that $m_j = 1$ for $1 \leq j \leq k$ by Lemma B.1, but this is not important at this stage.

which is an entire function, it follows that the function v defined by

$$v(z) = \hat{u}(z)/\mathcal{H}(z) = \hat{u}(z)\frac{\Gamma(z)\Xi(z)}{\det Q(z)} \quad \text{in } \mathbb{C} \tag{3.37}$$

is also an entire function.

It is clear that

$$\hat{u}(z) = v(z)\mathcal{H}(z) \quad \text{in } \mathbb{C}. \tag{3.38}$$

We consider the holomorphic function v restricted to $\mathcal{L}_m := \{z \in \mathbb{C};\ |\Re(z)| \le cm,\ -((2m+1)/(\sqrt{3}\,L))^3 \le \Im(z) \le ((2m+1)/(\sqrt{3}\,L))^3\}$ with large $m \in \mathbb{N}_*$. Using Proposition 2.8 to bound $\hat{u}$, and Lemma B.3 to bound $(\det Q(z))^{-1}$, we can bound v on $\partial\mathcal{L}_m$ (and thus also in the interior of $\mathcal{L}_m$) by

$$|v(z)| \le C_\varepsilon e^{(T+\varepsilon/2)((2m+1)/(\sqrt{3}\,L))^3} \quad \text{in } \mathcal{L}_m, \tag{3.39}$$

for all $\varepsilon > 0$, since, for large $|z|$,

$$|\Xi(z)| \le C|z|.$$

Note that the constant C_ε can be chosen independently of m. Here we use the fact

$$|\hat{u}(z)| \le Ce^{T|\Im(z)|} \quad \text{for } z \in \mathbb{C}.$$

On the other hand, applying Lemmas 3.3 and B.3 (2), we have

$$|v(z)| \le C_\varepsilon e^{(T+\varepsilon)|z|} \tag{3.40}$$

in $\{z \in \mathbb{C};\ |\Re(z)| \ge cm,\ -((2m+1)/(\sqrt{3}\,L))^3 \le \Im(z) \le ((2m+1)/(\sqrt{3}\,L))^3\}$.

Combining (3.39) and (3.40) yields

$$|v(z)| \le C_\varepsilon e^{(T+\varepsilon)|z|} \quad \text{in } \mathbb{C}. \tag{3.41}$$

Since $\mathcal{H}$ is a non-constant entire function, there exists $\gamma > 0$ such that

$$\mathcal{H}'(z + i\gamma) \ne 0 \quad \text{for all } z \in \mathbb{R}. \tag{3.42}$$

Fix such a γ and denote $\mathcal{H}_\gamma(z) = \mathcal{H}(z + i\gamma)$ for $z \in \mathbb{C}$.

Let us prove some asymptotics for $\mathcal{H}_\gamma$. Since $\sum_{j=1}^{3} \lambda_j = 0$, it follows from (2.6) that

$$\det Q = (\lambda_2 - \lambda_1)e^{-\lambda_3 L} + (\lambda_3 - \lambda_2)e^{-\lambda_1 L} + (\lambda_1 - \lambda_3)e^{-\lambda_2 L}.$$

We use the convention in Lemma 3.3. Thus, by Lemma 3.3, for fixed $\beta \ge 0$,

$$\begin{aligned}\mathcal{H}(z + i\beta) &= \frac{\det Q(z + i\gamma)}{\Xi(z + i\gamma)\Gamma(z + i\gamma)} \\ &= \kappa z^{-2/3-\sum_{i=1}^{k} m_j} e^{-\mu_1 L z^{1/3}}(1 + O(z^{-1/3})),\end{aligned} \tag{3.43}$$

where
$$\kappa = -\frac{1}{(\mu_2 - \mu_1)(\mu_1 - \mu_3)}.$$
We can also compute the asymptotic expansion of $\mathcal{H}'(z + i\beta)$, either by explicitly computing the asymptotic behavior of $\lambda_j'(z + i\beta)$ for large positive z (formally, one just needs to take the derivative of (3.43) with respect to z), or by using the Cauchy integral formula on the contour $\partial D(z, r)$ for some fixed r to justify differentiating (3.43). We get
$$\mathcal{H}'(z + i\beta) = -\frac{\mu_1 L}{3} z^{-2/3} \kappa z^{-2/3 - \sum_{i=1}^{k} m_j} e^{-\mu_1 L z^{1/3}} (1 + O(z^{-1/3})).$$
We then get
$$\lim_{z \in \mathbb{R},\, z \to +\infty} \mathcal{H}(z)|z|^{-2/3} / \mathcal{H}_\gamma'(z) = \alpha := 3e^{-i\pi/6}/L.$$
Similarly, we obtain
$$\lim_{z \in \mathbb{R},\, z \to -\infty} \mathcal{H}(z)|z|^{-2/3} / \mathcal{H}_\gamma'(z) = -\bar{\alpha}.$$
Moreover, we have
$$\begin{aligned} |\mathcal{H}(z)|z|^{-2/3} - \alpha \mathcal{H}_\gamma'(z)| &\le C|\mathcal{H}(z)|\,|z|^{-1} \\ &\le C|\mathcal{H}_\gamma'(z)|\,|z|^{-1/3} \quad \text{for large positive } z, \end{aligned} \tag{3.44}$$
and
$$\begin{aligned} \left|\mathcal{H}(z)|z|^{-2/3} + \bar{\alpha} \mathcal{H}_\gamma'(z)\right| &\le C|\mathcal{H}(z)|\,|z|^{-1} \\ &\le C|\mathcal{H}_\gamma'(z)|\,|z|^{-1/3} \quad \text{for large negative } z. \end{aligned} \tag{3.45}$$

Set
$$\hat{w}(z) = v(z)\mathcal{H}_\gamma'(z) = \hat{u}(z)\mathcal{H}_\gamma'(z)\mathcal{H}(z)^{-1}. \tag{3.46}$$
Then $\hat{w}$ is an entire function and satisfies Paley–Wiener's conditions for the interval $(-T - \varepsilon, T + \varepsilon)$ for all $\varepsilon > 0$ (see e.g. [41, 19.3 Theorem]). Indeed, this follows from the facts $|\hat{w}(z)| \le C_\varepsilon |v(z)| e^{\varepsilon|z|}$ for $z \in \mathbb{C}$ by Lemma 3.3, $|v(z)| \le C_\varepsilon e^{(T+\varepsilon)|z|}$ for $z \in \mathbb{C}$ by (3.41), $|\mathcal{H}_\gamma'(z) v(z)| = |\mathcal{H}_\gamma'(z)\mathcal{H}(z)^{-1}\hat{u}(z)| \le |\hat{u}(z)|$ for real z with large $|z|$, so that $\int_{\mathbb{R}} |\hat{w}|^2 < +\infty$.

We claim that[6]
$$\left| \int_0^L B(z, x)\, dx \right| \le \frac{C}{(|z| + 1)^{4/3}} \quad \text{for } z \in \mathbb{R}. \tag{3.47}$$
In fact, this inequality follows from Lemma 3.4 for large z, and from Lemma B.1 otherwise, since if z is a real solution of the equation $H(z) = 0$, which is simple by Lemma B.1, we have, by Lemma B.1 again,
$$\sum_{j=1}^{3} (e^{\lambda_{j+1} L} - e^{\lambda_j L}) e^{\lambda_{j+2} x} \overset{(B.2)}{=} 0.$$

[6]Recall that B was defined in (3.5).

From (3.42), (3.44), (3.45), and (3.47), we derive

$$\left| \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx \right| \le C|\hat{w}(z)|\,|\hat{w}(z-p)| \quad \text{for } z \in \mathbb{R}. \tag{3.48}$$

Note that, for $m \ge 1$,

$$\begin{aligned} \Big| \int_{|z|>m} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz - E|\alpha|^2 \int_{|z|>m} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz \Big| \\ \le \int_{|z|>m} \left| \hat{u}(z)\overline{\hat{u}(z-p)} \Big(\int_0^L B(z,x)\,dx - E|z|^{-4/3} \Big) \right| dz \\ + |E| \int_{|z|>m} \left| |\alpha|^2 \hat{w}(z)\overline{\hat{w}(z-p)} - |z|^{-4/3} \hat{u}(z)\overline{\hat{u}(z-p)} \right| dz. \end{aligned}$$

Using (3.44), (3.45), and Lemmas 3.1 and 3.4, we derive

$$\begin{aligned} \Big| \int_{|z|>m} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz - E|\alpha|^2 \int_{|z|>m} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz \Big| \\ \le C \int_{|z|>m} |\hat{w}(z)|\,|\hat{w}(z-p)||z|^{-1/3}\,dz. \end{aligned}$$

We deduce from (3.42) and (3.48) that

$$\begin{aligned} \Big| \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz - E|\alpha|^2 \int_{\mathbb{R}} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz \Big| \\ \le C \int_{|z|\le m} |\hat{w}(z)|\,|\overline{\hat{w}(z-p)}|\,dz + Cm^{-1/3} \int_{|z|>m} |\hat{w}(z)|\,|\hat{w}(z-p)|\,dz. \end{aligned}$$

Since, for $z \in \mathbb{R}$,

$$|\hat{w}(z)| \le C\|w\|_{L^1} = C\|w\|_{L^1(-T,T)} \le CT^{1/2}\|w\|_{L^2(\mathbb{R})},$$

we derive

$$\begin{aligned} \Big| \int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz - E|\alpha|^2 \int_{\mathbb{R}} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz \Big| \\ \le C \int_{-T}^{T} (Tm + m^{-1/3})|w|^2. \end{aligned}$$

Using the fact that

$$\int_{\mathbb{R}} \hat{w}(z)\overline{\hat{w}(z-p)}\,dz = \int_{\mathbb{R}} |w(t)|^2 e^{-itp}\,dt = \int_{-T}^{T} |w(t)|^2 e^{-itp}\,dt,$$

we obtain, by choosing $m = 1/T^{3/4}$,

$$\int_{\mathbb{R}} \hat{u}(z)\overline{\hat{u}(z-p)} \int_0^L B(z,x)\,dx\,dz = E|\alpha|^2 \int_{-T}^{T} |w(t)|^2 (1 + O(1)T^{1/4})\,dt.$$

The conclusion follows by noting that

$$\int_{\mathbb{R}} |w(t)|^2 \, dt = \int_{\mathbb{R}} |\hat{w}(z)|^2 \, dz \geq C \int_{\mathbb{R}} \frac{|\hat{u}(z)|^2}{1 + |z|^{4/3}} \, dz.$$

The proof is complete. ■

4. Useful estimates for linear KdV equations

In this section, we establish several results for linear KdV equations which will be used in the proof of Theorem 1.2. Our study of inhomogeneous KdV equations is based on three elements. The first one is the information on KdV equations explored previously. The second one is a connection between KdV equations and KdV-Burgers equations, as previously suggested in [13, 29]. The third one is estimates for KdV-Burgers equations with periodic boundary condition. This section contains two subsections: on inhomogeneous KdV-Burgers equations with periodic boundary condition and on inhomogeneous KdV equations.

4.1. On linear KdV-Burgers equations

In this section, we derive several estimates for solutions of linear KdV-Burgers equations using low regular data information. The main result of this section is the following result:

Lemma 4.1. *Let $L > 0$ and $f_1 \in L^1(\mathbb{R}_+; L^1(0, L))$ and $f_2 \in L^1(\mathbb{R}_+; W^{1,1}(0, L))$ be such that*

$$\int_0^L f_1(t, x) \, dx = 0 \quad \text{for a.e. } t > 0, \tag{4.1}$$

and

$$f_2(t, 0) = f_2(t, L) = 0 \quad \text{for a.e. } t > 0. \tag{4.2}$$

Set $f = f_1 + f_{2,x}$ and assume that $f \in L^1(\mathbb{R}_+; L^2(0, L))$. Let y be the unique solution in $C([0, +\infty); L^2(0, L)) \cap L^2_{\text{loc}}([0, +\infty); H^1(0, L))$, which is periodic in space, of the system

$$y_t(t, x) + 4y_x(t, x) + y_{xxx}(t, x) - 3y_{xx}(t, x) = f(t, x) \quad \text{in } (0, +\infty) \times (0, L), \tag{4.3}$$

and

$$y(t = 0, \cdot) = 0 \quad \text{in } (0, L). \tag{4.4}$$

We have, for $x \in [0, L]$,

$$\|y(\cdot, x)\|_{L^2(\mathbb{R}_+)} + \|y_x(\cdot, x)\|_{H^{-1/3}(\mathbb{R})} \leq C \|f\|_{L^1(\mathbb{R}_+ \times (0,L))}, \tag{4.5}$$

and

$$\|y(\cdot, x)\|_{H^{-1/3}(\mathbb{R})} + \|y_x(\cdot, x)\|_{H^{-2/3}(\mathbb{R})} + \|y\|_{L^2(\mathbb{R}_+; H^{-1}(0,L))} \leq C \|(f_1, f_2)\|_{L^1(\mathbb{R}_+ \times (0,L))}. \tag{4.6}$$

Assume that $f(t,\cdot) = 0$ *for* $t > T$. *We have, for all* $\delta > 0$ *and all* $t \geq T + \delta$,

$$|y_t(t,x)| + |y_x(t,x)| \leq C_\delta \|(f_1, f_2)\|_{L^1(\mathbb{R}_+ \times (0,L))} \quad \text{for } x \in [0, L]. \tag{4.7}$$

Here C (*resp.* C_δ) *denotes a positive constant depending only on* L (*resp.* L *and* δ).

Remark 4.2. Using the standard energy method, as for KdV equations, one can prove that if $f \in L^1(\mathbb{R}_+, L^2(0,L))$ with $\int_0^L f(t,x)\,dx = 0$ for a.e. $t > 0$ (this holds by (4.1) and (4.2)), then (4.3)–(4.4) has a unique solution in $C([0,+\infty); L^2(0,L)) \cap L^2([0,+\infty); H^1(0,L))$ which is periodic in space.

In the proof of Lemma 4.1, we use the following elementary estimate, which has its root in the work of Bourgain [14].

Lemma 4.3. *There exists a positive constant* C *such that, for* $j = 0, 1$ *and* $z \in \mathbb{R}$,[7]

$$\sum_{n \neq 0} \frac{|n|^j}{|z + 4n - n^3| + n^2} \leq \frac{C \ln(|z| + 2)}{(|z| + 2)^{(2-j)/3}}. \tag{4.8}$$

Proof. For $z \in \mathbb{R}$, let $k \in \mathbb{Z}$ be such that $k^3 \leq z < (k+1)^3$. It is clear that

$$\sum_{n \neq 0} \frac{|n|^j}{|z + 4n - n^3| + n^2} = \sum_{m+k \neq 0} \frac{|m+k|^j}{|z + 4(m+k) - (m+k)^3| + (m+k)^2}. \tag{4.9}$$

We split the sum into two parts, for $|m| \leq 2|k| + 2$ and for $|m| > 2|k| + 2$. Since $k^3 \leq z < (k+1)^3$, one can check that, for $m \in \mathbb{Z}$, $m + k \neq 0$, and $|m| \leq 2|k| + 2$,

$$|z + 4(m+k) - (m+k)^3| + |m+k|^2 \geq C(|m| + 1)(|k| + 2)^2,$$

and for $|m| \geq 2|k| + 2$,

$$|z + 4(m+k) - (m+k)^3| + |m+k|^2 \geq C|m|^3$$

(by considering $|k| \geq 10$ and $|k| < 10$). We deduce that

$$\begin{aligned}
\sum_{|m| \leq 2|k|+2,\, m+k \neq 0} & \frac{|m+k|^j}{|z + 4(m+k) - (m+k)^3| + (m+k)^2} \\
& \leq C \sum_{|m| \leq 2|k|+2} \frac{1}{(|k|+2)^{2-j}(|m|+1)} \\
& \leq \frac{C \ln(|k|+2)}{(|k|+2)^{2-j}},
\end{aligned} \tag{4.10}$$

[7] We recall that an absolutely convergent sum is none other than an integral with respect to the counting measure which is σ-finite. In the following, we will often exchange sums and integrals without comments, the justification being by the use of Fubini's theorem.

and

$$\sum_{|m|>2|k|+2} \frac{|m+k|^j}{|z+4(m+k)-(m+k)^3|+(m+k)^2} \leq C \sum_{|m|>2|k|+2} \frac{1}{|m|^{3-j}} \leq \frac{C}{(|k|+2)^{2-j}}. \tag{4.11}$$

Combining (4.9)–(4.11) yields (4.8). ∎

In what follows, for an appropriate function ζ defined in $\mathbb{R}_+ \times (0, L)$, we denote

$$\hat{\hat{\zeta}}(z,n) = \frac{1}{L}\int_0^L \hat{\zeta}(z,x) e^{-i2\pi n x/L}\, dx \quad \text{for } (z,n) \in \mathbb{R}\times\mathbb{Z}.$$

Recall that to define $\hat{\zeta}(z,x)$, we extend ζ by 0 for $t < 0$.

Proof of Lemma 4.1. For simplicity of notations we will assume that $L = 2\pi$. We establish (4.5)–(4.7) in Steps 1–3 below.

Step 1: Proof of (4.5). We first estimate $\|y(\cdot,x)\|_{L^2(\mathbb{R}_+)}$ for $x \in [0,L]$. From (4.3) and (4.4), we have

$$\hat{\hat{y}}(z,n) = \frac{\hat{\hat{f}}(z,n)}{i(z+4n-n^3)+3n^2} \quad \text{for } (z,n) \in \mathbb{R}\times(\mathbb{Z}\setminus\{0\}), \tag{4.12}$$

and

$$\hat{\hat{y}}(z,0) = 0 \quad \text{for } z \in \mathbb{R} \tag{4.13}$$

since $\int_0^L f(t,x)\,dx = 0$ for $t > 0$ by (4.1) and (4.2). By Plancherel's theorem, we obtain

$$\int_{\mathbb{R}_+} |y(t,x)|^2\, dt = \int_{\mathbb{R}} |\hat{y}(z,x)|^2\, dz \leq C \int_{\mathbb{R}} \left| \sum_{n\neq 0} \frac{|\hat{\hat{f}}(z,n)|}{|z+4n-n^3|+n^2} \right|^2 dz. \tag{4.14}$$

Since

$$|\hat{\hat{f}}(z,n)| \leq C\|f\|_{L^1(\mathbb{R}_+\times(0,L))}, \tag{4.15}$$

it follows from (4.14) that

$$\int_{\mathbb{R}_+} |y(t,x)|^2\, dt \leq C\|f\|^2_{L^1(\mathbb{R}_+\times(0,L))} \int_{\mathbb{R}} \left| \sum_{n\neq 0} \frac{1}{|z+4n-n^3|+n^2} \right|^2 dz. \tag{4.16}$$

Applying Lemma 4.3 with $j = 0$, we deduce from (4.16) that

$$\int_{\mathbb{R}_+} |y(t,x)|^2\, dt \leq C\|f\|^2_{L^1(\mathbb{R}_+\times(0,L))} \int_{\mathbb{R}} \frac{\ln^2(|z|+2)}{(|z|+2)^{4/3}}\, dz,$$

which yields

$$\|y(\cdot,x)\|_{L^2} \leq C\|f\|_{L^1(\mathbb{R}_+\times(0,L))}. \tag{4.17}$$

We next estimate $\|y_x(\cdot,x)\|_{H^{-1/3}(\mathbb{R}_+)}$ for $x \in [0,L]$. By (4.12), (4.13), and (4.15), we have

$$\|y_x(\cdot,x)\|^2_{H^{-1/3}(\mathbb{R}_+)} \le C\|f\|^2_{L^1(\mathbb{R}_+\times(0,L))} \int_{\mathbb{R}} \frac{1}{(1+|z|^2)^{1/3}} \left| \sum_{n\neq 0} \frac{|n|}{|z+4n-n^3|+n^2} \right|^2 dz. \tag{4.18}$$

Applying Lemma 4.3 with $j=1$, we deduce from (4.18) that

$$\|y_x(\cdot,x)\|^2_{H^{-1/3}(\mathbb{R}_+)} \le C\|f\|^2_{L^1(\mathbb{R}_+\times(0,L))} \int_{\mathbb{R}} \frac{\ln^2(|z|+2)}{(|z|+2)^{4/3}}\, dz,$$

which yields

$$\|y_x(\cdot,x)\|_{H^{-1/3}(\mathbb{R})} \le C\|f\|_{L^1(\mathbb{R}_+\times(0,L))}. \tag{4.19}$$

Assertion (4.5) now follows from (4.17) and (4.19).

Step 2: Proof of (4.6). By Step 1, without loss of generality, one might assume that $f_1 = 0$. The proof of the inequality $\|y(\cdot,x)\|_{H^{-1/3}} \le C\|f_2\|_{L^1(\mathbb{R}_+\times(0,L))}$ is similar to the one of (4.19) and is omitted.

To prove

$$\|y_x(\cdot,x)\|_{H^{-2/3}(\mathbb{R})} \le C\|f_2\|_{L^1(\mathbb{R}_+\times(0,L))}, \tag{4.20}$$

we proceed as follows. For $z \in \mathbb{R}$,

$$\hat{y}_x(z,x) = -\frac{1}{L}\int_0^L \hat{f}_2(z,\xi) \sum_{n\neq 0} \frac{n^2 e^{in(x-\xi)}}{i(z+4n-n^3)+3n^2}\, d\xi. \tag{4.21}$$

We have, for some large positive constant c,

$$\left| \sum_{|n|\ge c(|z|+1)} \frac{n^2 e^{in(x-\xi)}}{i(z+4n-n^3)+3n^2} + \sum_{|n|\ge c(|z|+1)} \frac{e^{in(x-\xi)}}{in} \right| \le C \sum_{|n|\ge c(|z|+1)} \frac{1}{|n|^2} \le \frac{C}{|z|+1},$$

$$\left| \sum_{0<|n|\le c(|z|+1)} \frac{e^{in(x-\xi)}}{in} \right| \le C\ln(|z|+2),$$

and, as in (4.10) in the proof of Lemma 4.3,

$$\left| \sum_{0<|n|\le c(|z|+1)} \frac{n^2 e^{in(x-\xi)}}{i(z+4n-n^3)+3n^2} \right| \le C\ln(|z|+2).$$

It follows that

$$\left| \sum_{n\neq 0} \frac{n^2 e^{in(x-\xi)}}{i(z+4n-n^3)+3n^2} + \sum_{n\neq 0} \frac{e^{in(x-\xi)}}{in} \right| \le \frac{C}{|z|+1} + C\ln(|z|+2). \tag{4.22}$$

Since

$$\sum_{n\neq 0} \frac{e^{in\xi'}}{in} = -\xi' + \pi \quad \text{for } \xi' \in (0, 2\pi),$$

and

$$\|y_x(\cdot, x)\|^2_{H^{-2/3}(\mathbb{R})} = \int_{\mathbb{R}} \frac{|\hat{y}_x(z, x)|^2}{(1 + |z|^2)^{2/3}}\, dz,$$

assertion (4.20) follows from (4.21) and (4.22).

We next find that the estimate

$$\|y\|_{L^2(\mathbb{R}_+; H^{-1}(0,L))} \leq C\|f_2\|_{L^1(\mathbb{R}_+\times(0,L))}$$

follows from

$$\|y\|^2_{L^2(\mathbb{R}_+; H^{-1}(0,L))} \leq C \int_{\mathbb{R}} \sum_{n\neq 0} \left| \frac{\hat{\hat{f}}_2(z, n)}{|i(z + 4n - n^3)| + 3n^2} \right|^2 dz$$

and Lemma 4.3. The proof of Step 2 is complete.

Step 3: Proof of (4.7). For simplicity of presentation, we will assume that $f_1 = 0$. We have the following representation for the solution:

$$y(t, x) = \sum_{n\neq 0} e^{inx} \int_0^t e^{-(i(4n-n^3)+3n^2)(t-\tau)} \left(\frac{in}{L} \int_0^L f_2(\tau, \xi) e^{-in\xi}\, d\xi \right) d\tau. \tag{4.23}$$

Let $\mathbb{1}_A$ denote the characteristic function of a set A in $\mathbb{R}$. Assertion (4.7) then follows easily from (4.23) by noting that, for $t \geq T + \delta$,

$$\sum_{n\neq 0} \int_0^t |n|^{10} e^{-3n^2(t-\tau)} \mathbb{1}_{\{\tau<T\}}\, d\tau < C_\delta.$$

The proof is complete. ■

4.2. *On linear KdV equations*

In this section, we derive various results on linear KdV equations using low regularity data information. These result will be used in the proof of Theorem 1.2. We begin with

Lemma 4.4. *Let* $h = (h_1, h_2, h_3) \in H^{1/3}(\mathbb{R}_+) \times H^{1/3}(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$, *and let* $y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\mathrm{loc}}([0, +\infty); H^1(0, L))$ *be the unique solution of the system*

$$\begin{cases} y_t(t, x) + y_x(t, x) + y_{xxx}(t, x) = 0 & \textit{in } (0, +\infty) \times (0, L), \\ y(t, x = 0) = h_1(t),\ y(t, x = L) = h_2(t),\ y_x(t, x = L) = h_3(t) & \textit{in } (0, +\infty), \end{cases} \tag{4.24}$$

and

$$y(t = 0, \cdot) = 0 \quad \textit{in } (0, L). \tag{4.25}$$

We have, for $T > 0$,

$$\|y\|_{L^2((0,T)\times(0,L))} \le C_{T,L}(\|(h_1,h_2)\|_{L^2(\mathbb{R}_+)} + \|h_3\|_{H^{-1/3}(\mathbb{R})}), \tag{4.26}$$

$$\|y\|_{L^2((0,T);H^{-1}(0,L))} \le C_{T,L}(\|(h_1,h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})}), \tag{4.27}$$

for some positive constant $C_{T,L}$ independent of h.

Here and in what follows, $H^{-1}(0,L)$ is the dual space of $H_0^1(0,L)$ with the corresponding norm.

Proof. By the linearity of the system and the uniqueness of solutions, it suffices to consider the three cases $(h_1,h_2,h_3) = (0,0,h_3)$, $(h_1,h_2,h_3) = (h_1,0,0)$, and $(h_1,h_2,h_3) = (0,h_2,0)$ separately.

We first consider the case $(h_1,h_2,h_3) = (0,0,h_3)$. Making a truncation, without loss of generality, one can assume that $h_3 = 0$ for $t > 2T$. Let $g_3 \in C^1(\mathbb{R})$ be such that $\operatorname{supp} g_3 \subset [T,3T]$, and if z is a real solution of the equation $\det Q(z)\Xi(z) = 0$ of order m then z is also a real zero of order m of $\hat{h}_3(z) - \hat{g}_3(z)$, and

$$\|g_3\|_{H^{-1/3}(\mathbb{R})} \le C_{T,L}\|h_3\|_{H^{-2/3}(\mathbb{R})}.$$

The construction of g_3, inspired by the moment method (see e.g. [45]), can be done as follows. Set $\eta(t) = e^{-1/(t^2-(T)^2)}\mathbb{1}_{|t|<T}$ for $t \in \mathbb{R}$. Assume that $z_1,\dots,z_k$ are real, distinct solutions of the equation $\det Q(z)\Xi(z) = 0$, and $m_1,\dots,m_k$ are their respective orders (the number of real solutions of the equation $\det Q(z)\Xi(z) = 0$ is finite by Lemma B.1 and in fact they are simple; nevertheless, we ignore this point and present a proof without using this information). Set, for $z \in \mathbb{C}$,

$$\zeta(z) = \sum_{i=1}^{k}\Big(\hat{\eta}(z-z_i)\prod_{\substack{j=1\\ j\neq i}}^{k}(z-z_j)^{m_j}\Big(\sum_{l=0}^{m_i} c_{i,l}(z-z_i)^l\Big)\Big),$$

where $c_{i,l} \in \mathbb{C}$ is chosen such that

$$\frac{d^l}{dz^l}(e^{2iTz}\zeta(z))\Big|_{z=z_i} = \frac{d^l}{dz^l}\hat{h}_3(z_i) \quad \text{for } 0 \le l \le m_i,\ 1 \le i \le k.$$

This can be done because $\hat{\eta}(0) \neq 0$. Since

$$|\hat{\eta}(z)| \le Ce^{T|\Im(z)|},$$

and, by [45, Lemma 4.3],

$$|\hat{\eta}(z)| \le C_1 e^{-C_2|z|^{1/2}} \quad \text{for } z \in \mathbb{R},$$

using Paley–Wiener's theorem one can prove that ζ is the Fourier transform of a function ψ of class C^1; moreover, ψ has support in $[-T,T]$. Set, for $z \in \mathbb{C}$,

$$g_3(t) = \psi(t+2T).$$

Using the fact $\hat{g}_3(z) = e^{i2Tz}\zeta(z)$, one can check that $\hat{g}_3 - \hat{h}_3$ has zeros $z_1, \dots, z_k$ of respective orders $m_1, \dots, m_k$. One can check that

$$\|\psi\|_{C^1} \le C_{T,L} \sum_{i=1}^{k} \sum_{l=0}^{m_i} \left| \frac{d^l}{dz^l} \hat{h}_3(z_i) \right|,$$

which yields

$$\|\psi\|_{C^1} \le C_{T,L} \|h_3\|_{H^{-2/3}(\mathbb{R})}.$$

The required properties of g_3 follow.

By considering the solution corresponding to $h_3 - g_3$, without loss of generality one can assume that if z is a real solution of order m of the equation $\det Q(z)\Xi(z) = 0$ then z is also a real zero of order at least m of $\hat{h}_3(z)$. This fact is assumed from now on.

We now establish (4.26). We have, by Lemma 2.4,

$$\hat{y}(z,x) = \frac{\hat{h}_3(z)}{\det Q} \sum_{j=1}^{3} (e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}) e^{\lambda_j x} \quad \text{for a.e. } x \in (0,L). \tag{4.28}$$

From the assumption on h_3, we have, for $z \in \mathbb{R}$ and $|z| \le \gamma$,

$$\left| \frac{\hat{h}_3(z)}{\det Q(z)} \sum_{j=1}^{3} (e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}) e^{\lambda_j x} \right| \le C_{T,\gamma} \|h_3\|_{H^{-2/3}(\mathbb{R})}, \tag{4.29}$$

and, by Lemma 3.3, for $z \in \mathbb{R}$ with $|z| \ge \gamma$ with sufficiently large γ,

$$\left| \frac{1}{\det Q} \sum_{j=1}^{3} (e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}) e^{\lambda_j x} \right| \le \frac{C}{(1+|z|)^{1/3}}. \tag{4.30}$$

Combining (4.29) and (4.30) yields

$$\|\hat{y}\|_{L^2(\mathbb{R}\times(0,L))} \le C_T \|h_3\|_{H^{-1/3}(\mathbb{R})},$$

which is (4.26) when $(h_1, h_2, h_3) = (0, 0, h_3)$.

We next deal with (4.27). The proof of (4.27) is similar to the one of (4.26). One just notes that, instead of (4.30), for $z \in \mathbb{R}$ with $|z| \ge \gamma$ with sufficiently large γ, we have

$$\left\| \frac{1}{\det Q} \sum_{j=1}^{3} (e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}) e^{\lambda_j x} \right\|_{H^{-1}(0,L)} \le \frac{C}{(1+|z|)^{2/3}}. \tag{4.31}$$

The details are omitted.

The proof in the case $(h_1, h_2, h_3) = (h_1, 0, 0)$ or in the case $(h_1, h_2, h_3) = (0, h_2, 0)$ is similar. We only mention here that the solution corresponding to $(h_1, 0, 0)$ is given by

$$\hat{y}(z,x) = \frac{\hat{h}_1(z)}{\det Q} \sum_{j=1}^{3} (\lambda_{j+2} - \lambda_{j+1}) e^{\lambda_j (x-L)} \quad \text{for a.e. } x \in (0,L),$$

and the solution corresponding to $(0, h_2, 0)$ is given by

$$\hat{y}(z,x) = \frac{\hat{h}_2(z)}{\det Q} \sum_{j=1}^{3} (\lambda_{j+1} e^{\lambda_{j+1} L} - \lambda_{j+2} e^{\lambda_{j+2} L}) e^{\lambda_j x} \quad \text{for a.e. } x \in (0, L).$$

The details are left to the reader. ■

Remark 4.5. The estimates in Lemma 4.4 are in the spirit of the well-posedness results due to Bona et al. [13] (see also [12]) but quite different. The setting of Lemma 4.4 is below the limiting case in [13], which was not investigated in their work.

We next establish a variant of Lemma 4.4 for inhomogeneous KdV systems.

Lemma 4.6. *Let $L > 0$ and $T > 0$. Let $h = (h_1, h_2, h_3) \in H^{1/3}(\mathbb{R}_+) \times H^{1/3}(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$, $f_1 \in L^1((0,T) \times (0,L))$, and $f_2 \in L^1((0,T); W^{1,1}(0,L))$ with*

$$f_2(t,0) = f_2(t,L) = 0. \tag{4.32}$$

Set $f = f_1 + f_{2,x}$ and assume that $f \in L^1(\mathbb{R}_+; L^2(0,L))$. Let $y \in C([0,+\infty); L^2(0,L)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^1(0,L))$ be the unique solution of the system

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = f(t,x) & \text{in } (0,+\infty) \times (0,L), \\ y(t,x=0) = h_1(t),\ y(t,x=L) = h_2(t),\ y_x(t,x=L) = h_3(t) & \text{in } (0,+\infty), \end{cases} \tag{4.33}$$

and

$$y(t=0,\cdot) = 0 \quad \text{in } (0,L).$$

We have

$$\|y\|_{L^2((0,T)\times(0,L))} \le C_T \big(\|(h_1,h_2)\|_{L^2(\mathbb{R}_+)} + \|h_3\|_{H^{-1/3}(\mathbb{R})} + \|f\|_{L^1(\mathbb{R}_+ \times (0,L))} \big), \tag{4.34}$$

and

$$\|y\|_{L^2((0,T);H^{-1}(0,L))} \le C_T \big(\|(h_1,h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})} + \|(f_1,f_2)\|_{L^1(\mathbb{R}_+ \times (0,L))} \big). \tag{4.35}$$

Assume in addition that $h(t,\cdot) = 0$ and $f(t,\cdot) = 0$ for $t \ge T_1$ for some $0 < T_1 < T$. Then, for any $\delta > 0$ and $T_1 + \delta \le t \le T$, we have

$$|y_t(t,x)| + |y_x(t,x)| \le C_{T,T_1,\delta} \big(\|(h_1,h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})} + \|(f_1,f_2)\|_{L^1(\mathbb{R}_+ \times (0,L))} \big). \tag{4.36}$$

Here C_T and $C_{T,T_1,\delta}$ denote positive constants independent of h and f.

Proof. The proof is based on a connection between KdV equations and KdV-Burgers equations. Set $v(t,x) = e^{-2t+x} y(t,x)$, which is equivalent to $y(t,x) = e^{2t-x} v(t,x)$.

Then

$$\begin{aligned} y_t(t,x) &= (2v(t,x) + v_t(t,x))e^{2t-x}, \\ y_x(t,x) &= (-v(t,x) + v_x(t,x))e^{2t-x}, \\ y_{xxx}(t,x) &= (v_{xxx}(t,x) - 3v_{xx}(t,x) + 3v_x(t,x) - v(t,x))e^{2t-x}. \end{aligned}$$

Hence, if y satisfies the equation

$$y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = f(t,x) \quad \text{in } \mathbb{R}_+ \times (0,L),$$

then

$$v_t(t,x) + 4v_x(t,x) + v_{xxx}(t,x) - 3v_{xx}(t,x) = f(t,x)e^{-2t+x} \quad \text{in } \mathbb{R}_+ \times (0,L).$$

Set, in $\mathbb{R}_+ \times (0,L)$,

$$\psi(t,x) = \psi(t) := \frac{1}{L}\int_0^L f(t,\xi)e^{-2t+\xi}\,d\xi \quad \text{and} \quad g(t,x) := f(t,x)e^{-2t+x} - \psi(t,x). \tag{4.37}$$

Then

$$\int_0^L g(t,x)\,dx = 0.$$

Let $y_1 \in C([0,+\infty); L^2(0,L)) \cap L^2_{\text{loc}}([0,+\infty); H^1(0,L))$ be the unique solution which is periodic in space of the system

$$y_{1,t}(t,x) + 4y_{1,x}(t,x) + y_{1,xxx}(t,x) - 3y_{1,xx}(t,x) = g(t,x) \quad \text{in } (0,+\infty)\times(0,L), \tag{4.38}$$

$$y_1(t=0,\cdot) = 0 \quad \text{in } (0,L). \tag{4.39}$$

We have, by (4.32),

$$g(t,x) = f_1(t,x)e^{-2t+x} + f_{2,x}(t,x)e^{-2t+x} - \psi(t,x), \tag{4.40}$$

$$\psi(t,x) = \frac{1}{L}\int_0^L f_1(t,\xi)e^{-2t+\xi}\,d\xi - \frac{1}{L}\int_0^L f_2(t,\xi)e^{-2t+\xi}\,d\xi. \tag{4.41}$$

Applying Lemma 4.1, we have

$$\|y_1(\cdot,x)\|_{L^2(\mathbb{R}_+)} + \|y_{1,x}(\cdot,x)\|_{H^{-1/3}(\mathbb{R})} \le C\|g\|_{L^1(\mathbb{R}_+\times(0,L))}$$

which yields, by (4.37),

$$\|y_1(\cdot,x)\|_{L^2(\mathbb{R}_+)} + \|y_{1,x}(\cdot,x)\|_{H^{-1/3}(\mathbb{R})} \le C\|f\|_{L^1(\mathbb{R}_+\times(0,L))}. \tag{4.42}$$

Similarly, by noting $f_{2,x}(t,x)e^{-2t+x} = (f_2(t,x)e^{-2t+x})_x - f_2(t,x)e^{-2t+x}$, we get

$$\|y_1(\cdot,x)\|_{H^{-1/3}(\mathbb{R})} + \|y_{1,x}(\cdot,x)\|_{H^{-2/3}(\mathbb{R})} \le C\|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))}. \tag{4.43}$$

Applying Lemma 4.1 again, we obtain

$$|y_{1,x}(t,x)| + |y_{1,t}(t,x)| \leq C_{T,T_1,\delta} \|(f_1, f_2)\|_{L^1(\mathbb{R}_+\times(0,L))} \quad \text{for } T_1 + \delta/2 \leq t \leq T. \tag{4.44}$$

if $f = 0$ for $t \geq T_1$.

Fix $\varphi \in C(\mathbb{R})$ such that $\varphi = 1$ for $|t| \leq T$ and $\varphi = 0$ for $|t| > 2T$. Let $y_2 \in C([0,+\infty); L^2(0,L)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^1(0,L))$ be the unique solution of the system

$$\begin{cases} y_{2,t}(t,x) + y_{2,x}(t,x) + y_{2,xxx}(t,x) = \varphi(t)\psi(t,x) & \text{in } (0,+\infty)\times(0,L), \\ y_2(t,x=0) = h_1(t) - \varphi(t)e^{2t}y_1(t,0) & \text{in } (0,+\infty), \\ y_2(t,x=L) = h_2(t) - \varphi(t)e^{2t-L}y_1(t,L) & \text{in } (0,+\infty), \\ y_{2,x}(t,x=L) = h_3(t) - \varphi(t)(e^{2t-\cdot}y_1(t,\cdot))_x(t,L) & \text{in } (0,+\infty), \end{cases}$$

and

$$y_2(t=0,\cdot) = 0 \quad \text{in } (0,L).$$

Using (4.40) and applying Lemma 4.4 to y_2, from (4.42) we have

$$\|y_2\|_{L^2((0,T)\times(0,L))} \leq C_T\big(\|(h_1,h_2)\|_{L^2(\mathbb{R}_+)} + \|h_3\|_{H^{-1/3}(\mathbb{R})} + \|f\|_{L^1(\mathbb{R}_+\times(0,L))}\big), \tag{4.45}$$

and from (4.43), we obtain

$$\begin{aligned} &\|y_2\|_{L^2((0,T);H^{-1}(0,L))} \\ &\quad \leq C_T\big(\|(h_1,h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})} + \|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))}\big). \end{aligned} \tag{4.46}$$

One can verify that $y_1 + y_2$ and y satisfy the same system for $0 \leq t \leq T$ and they are in the space $C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$. By the well-posedness of the KdV system, one has

$$y = y_1 + y_2 \quad \text{in } (0,T)\times(0,L).$$

Combining (4.42) and (4.45) yields (4.34), and combining (4.43) and (4.46) yields (4.35). Combining (4.44) and (4.45) gives, for some $T_1 + \delta/2 \leq \tau \leq T_1 + 3\delta/4$,

$$\begin{aligned} &\|y(\tau,\cdot)\|_{H^{-1}(0,L)} \\ &\quad \leq C_{T,T_1,\delta}\big(\|(h_1,h_2)\|_{H^{-1/3}(\mathbb{R})} + \|h_3\|_{H^{-2/3}(\mathbb{R})} + \|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))}\big), \end{aligned} \tag{4.47}$$

and assertion (4.36) follows by the standard C^∞ smoothness property of solutions of the linear KdV system (4.33). The proof is complete. ■

Remark 4.7. One can also check (4.47) by using a variant of (4.7) in Lemma 4.1 in which $f = 0$ but a non-zero initial condition is considered.

5. Small time local null-controllability of the KdV system

The main result of this section is the following, which implies in particular Theorem 1.2.

Theorem 5.1. *Let $L > 0$, and $k, l \in \mathbb{N}_*$. Set*

$$p = \frac{(2k + l)(k - l)(2l + k)}{3\sqrt{3}\,(k^2 + kl + l^2)^{3/2}}. \tag{5.1}$$

Assume that

$$L = 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, \tag{5.2}$$

$$2k + l \notin 3\mathbb{N}_*. \tag{5.3}$$

Let Ψ be defined in (6.8), where

$$\eta_1 = -\frac{2\pi i}{3L}(2k + l), \quad \eta_2 = \eta_1 + \frac{2\pi i}{L}k, \quad \eta_3 = \eta_2 + \frac{2\pi i}{L}l, \tag{5.4}$$

and E is given by (3.10). There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, all[8] *$0 < T < T_*/2$ and for all solutions $y \in C([0, +\infty); H^2(0, L)) \cap L^2_{\text{loc}}([0, +\infty); H^3(0, L))$ of*

$$\begin{cases} y_t(t, x) + y_x(t, x) + y_{xxx}(t, x) + y y_x(t, x) = 0 & \text{in } (0, +\infty) \times (0, L), \\ y(t, x = 0) = y(t, x = L) = 0 & \text{in } (0, +\infty), \\ y_x(t, x = L) = u(t) & \text{in } (0, \infty), \\ y(0, \cdot) = y_0(x) := \varepsilon \Psi(0, \cdot) & \text{in } (0, L), \end{cases} \tag{5.5}$$

with $u \in H^{2/3}(\mathbb{R}_+)$, $\|u\|_{H^{2/3}(\mathbb{R})} < \varepsilon_0$, $u(0) = 0$, and $\operatorname{supp} u \subset [0, T]$, we have

$$y(T, \cdot) \neq 0.$$

Remark 5.2. With the choices of p and L in Theorem 5.1, the function $\Psi(t, x)$ given in Corollary 3.7 satisfies the linear KdV system as in [18], i.e.,

$$\Psi_t(t, x) + \Psi_{xxx}(t, x) + \Psi_x(t, x) = 0 \quad \text{in } \mathbb{R}_+ \times (0, L), \tag{5.6}$$

$$\Psi(t, 0) = \Psi(t, L) = \Psi_x(t, 0) = \Psi_x(t, L) = 0 \quad \text{in } \mathbb{R}_+. \tag{5.7}$$

This property can be rechecked using the fact η_1, η_2, η_3 are the roots of $\eta^3 + \eta - ip = 0$.

We first show that E defined by (3.10) with η_j given in (5.4) and with p as in (5.1) is not zero if (5.3) holds. More precisely, we have

Lemma 5.3. *Let $k, l \in \mathbb{N}_*$ and let E be given by (3.10) with η_j in (5.4) and with p as in (5.1). Assume that (6.2) holds. We have*

$$E = \frac{40\pi^3}{3L^3}(e^{\eta_1 L} - 1) i k l (k + l).$$

Consequently,

$$E \neq 0 \quad \text{provided that (5.3) holds.}$$

[8] T_* is the constant in Corollary 3.7 with p, η_j, and L given previously. Note that $E \neq 0$ by Lemma 5.3 below.

Proof. With $\gamma_j = L\eta_j/(2\pi i)$, we have

$$\gamma_1 = -\frac{2k+l}{3}, \quad \gamma_2 = \frac{k-l}{3}, \quad \gamma_3 = \frac{k+2l}{3}.$$

It follows that

$$\begin{aligned}\frac{L^3}{(2\pi i)^3}\sum_{j=1}^{3}\eta_{j+2}^2(\eta_{j+1}-\eta_j) &= \sum_{j=1}^{3}\gamma_{j+2}^2(\gamma_{j+1}-\gamma_j) = \gamma_3^2 k + \gamma_1^2 l - \gamma_2^2(k+l)\\ &= (\gamma_3^2-\gamma_2^2)k - (\gamma_2^2-\gamma_1^2)l = (k+l)kl,\end{aligned}$$

which yields

$$\sum_{j=1}^{3}\eta_{j+2}^2(\eta_{j+1}-\eta_j) = -8\pi^3 ikl(k+l)/L^3.$$

We also have

$$\begin{aligned}\sum_{j=1}^{3}\frac{\eta_{j+1}-\eta_j}{\eta_{j+2}} &= \sum_{j=1}^{3}\frac{\gamma_{j+1}-\gamma_j}{\gamma_{j+2}}\\ &= \frac{3k}{k+2l} - \frac{3l}{2k+l} - \frac{3(k+l)}{k-l}\\ &= -\frac{27kl(k+l)}{(k+2l)(2k+l)(k-l)}.\end{aligned}$$

We then have, by (3.10),

$$E = \frac{1}{3}(e^{\eta_1 L}-1)\left(\frac{16\pi^3 i}{3L^3}kl(k+l) + \frac{27ipkl(k+l)}{(k-l)(k+2l)(2l+k)}\right). \tag{5.8}$$

From (5.1) and (6.2), we obtain

$$\frac{p}{(k-l)(k+2l)(2l+k)} = \left(\frac{2\pi}{3L}\right)^3.$$

We deduce from (5.8) that

$$E = \frac{40\pi^3}{3L^3}(e^{\eta_1 L}-1)ikl(k+l).$$

The proof is complete. ∎

Before giving the proof of Theorem 5.1, we state and prove new estimates for the nonlinear KdV system (1.1)–(1.2), which play a role in the proof of Theorem 5.1.

Lemma 5.4. *Let $L, T > 0$. There exists a constant $\varepsilon_0 > 0$ depending on L and T such that for $y_0 \in L^2(0, L)$ and $u \in L^2(\mathbb{R}_+)$ with*

$$\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)} \le \varepsilon_0,$$

the unique solution $y \in C([0,+\infty); L^2(0,L)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^1(0,L))$ *of the system*

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) + y(t,x)y_x(t,x) = 0 & \text{in } (0,+\infty)\times(0,L), \\ y(t,x=0) = y(t,x=L) = 0 & \text{in } (0,+\infty), \\ y_x(t,x=L) = u(t) & \text{in } (0,\infty), \end{cases}$$

with $y(0,\cdot) = y_0$, *satisfies*

$$\|y\|_{L^2((0,T)\times(0,L))} \le C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})}), \tag{5.9}$$

$$\|y\|_{L^2((0,T);H^{-1}(0,L))} \le C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}), \tag{5.10}$$

where C *is a positive constant depending only on* T *and* L.

Proof. We have (see e.g. [24, Proposition 14]), for ε_0 small,

$$\|y_x\|_{L^2((0,T)\times(0,L))} \le C_T(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)}),$$

which yields

$$\|y_x\|_{L^2((0,T)\times(0,L))} \le C\varepsilon_0. \tag{5.11}$$

Set

$$f(t,x) = -y(t,x)\partial_x y(t,x).$$

The Cauchy–Schwarz inequality and (5.11) yield

$$\|f\|_{L^1(\mathbb{R}_+\times(0,L))} \le C\varepsilon_0\|y\|_{L^2(\mathbb{R}_+\times(0,L))}.$$

Applying Lemma 4.6, and more precisely (4.34), we have

$$\|y\|_{L^2(\mathbb{R}_+\times(0,L))} \le C\varepsilon_0\|y\|_{L^2(\mathbb{R}_+\times(0,L))} + C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})}).$$

By choosing ε_0 sufficiently small, the first term of the RHS can be absorbed by the LHS and assertion (5.9) follows.

To prove (5.10), one notes that

$$\begin{aligned} \|y^2\|_{L^1((0,T)\times(0,L))} &\le C\|y\|_{L^2((0,T);H^{-1}(0,L))}\|y\|_{L^2((0,T);H^1(0,L))} \\ &\overset{(5.11)}{\le} C\varepsilon_0\|y\|_{L^2((0,T);H^{-1}(0,L))}. \end{aligned}$$

By Lemma 4.6 (this time (4.35)), we obtain

$$\|y\|_{L^2((0,T);H^{-1}(0,L))} \le C\varepsilon_0\|y\|_{L^2((0,T);H^{-1}(0,L))} + C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}).$$

By choosing ε_0 sufficiently small, the first term of the RHS can be absorbed by the LHS and assertion (5.10) follows. ∎

Proof of Theorem 5.1. By Lemma 5.3, the constant E is not 0. Let ε_0 be a small positive constant, which depends only on k and l and is determined later. We prove Theorem 5.1 by contradiction. Assume that there exists a solution $y \in C([0,+\infty); H^2(0,L)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^3(0,L))$ of (5.5) with $y(t,\cdot) = 0$ for $t \geq T$, for some $u \in H^{2/3}(0,+\infty)$, some $0 < \varepsilon < \varepsilon_0$, and some $0 < T < T_*/2$ with $\|u\|_{H^{2/3}(\mathbb{R}_+)} < \varepsilon_0$, $u(0) = 0$, and $\operatorname{supp} u \subset [0,T]$.

For ε_0 small, we have (see e.g. [24, Proposition 14])

$$\|y\|_{L^2((0,T);H^1(0,L))} \leq C(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)}). \tag{5.12}$$

Set

$$y_1(t,x) = y(t,x) - c\int_0^L y(t,\eta)\Psi(t,\eta)\,d\eta\,\Psi(t,x), \tag{5.13}$$

with $c^{-1} := \int_0^L |\Psi(0,\eta)|^2\,d\eta$. Since $y_0(x) = \epsilon\Psi(0,x)$, this choice of c ensures that $y_1(0,\cdot) = 0$ in $(0,L)$. Then $y_1 \in C([0,+\infty); L^2(0,L)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^1(0,L))$ is the solution of

$$\begin{cases} y_{1,t}(t,x) + y_{1,x}(t,x) + y_{1,xxx}(t,x) + f(t,x) = 0 & \text{in } (0,+\infty)\times(0,L),\\ y_1(t,x=0) = y_1(t,x=L) = 0 & \text{in } (0,+\infty),\\ y_{1,x}(t,x=L) = u(t) & \text{in } (0,+\infty),\\ y_1(0,\cdot) = 0, \end{cases}$$

where

$$f(t,x) = f_1(t,x) + f_{2,x}(t,x),$$

with

$$f_1(t,x) = -c\int_0^L yy_x(t,\eta)\Psi(t,\eta)\,d\eta\,\Psi(t,x) = \frac{c}{2}\int_0^L y^2(t,\eta)\Psi_x(t,\eta)\,d\eta\,\Psi(t,x),$$
$$f_2(t,x) = \frac{1}{2}y^2(t,x).$$

By Lemma 5.4, we have

$$\|y\|_{L^2((0,T)\times(0,L))} \leq C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})}), \tag{5.14}$$

$$\|y\|_{L^2((0,T);H^{-1}(0,L))} \leq C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}). \tag{5.15}$$

From the definition of y_1 in (5.13), and (5.15), we obtain

$$\|y_1\|_{L^2((0,T);H^{-1}(0,L))} \leq C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}). \tag{5.16}$$

Let $y_2 \in C([0,+\infty); L^2(0,L)) \cap L^2_{\mathrm{loc}}([0,+\infty); H^1(0,L))$ be the unique solution of

$$\begin{cases} y_{2,t}(t,x) + y_{2,x}(t,x) + y_{2,xxx}(t,x) = -f(t,x) & \text{in } (0,+\infty)\times(0,L),\\ y_2(t,x=0) = y_2(t,x=L) = 0 & \text{in } (0,+\infty),\\ y_{2,x}(t,x=L) = 0 & \text{in } (0,+\infty),\\ y_2(0,\cdot) = 0, \end{cases}$$

and let $y_3 \in C([0,+\infty); L^2(0,L)) \cap L^2_{\text{loc}}([0,+\infty); H^1(0,L))$ be the unique solution of

$$\begin{cases} y_{3,t}(t,x) + y_{3,x}(t,x) + y_{3,xxx}(t,x) = 0 & \text{in } (0,+\infty)\times(0,L), \\ y_3(t,x=0) = y_3(t,x=L) = 0 & \text{in } (0,+\infty), \\ y_{3,x}(t,x=L) = u(t) & \text{in } (0,+\infty), \\ y_3(0,\cdot) = 0. \end{cases}$$

Then

$$y_1 = y_2 + y_3.$$

There exists $u_4 \in L^2(0,+\infty)$ such that $\operatorname{supp} u_4 \subset [2T_*/3, T_*]$,

$$\|u_4\|_{L^2(0,+\infty)} \le C \|y_3(2T_*/3,\cdot)\|_{L^2(2T_*/3,T_*)},$$

and

$$y_4(T_*,\cdot) = 0,$$

where $y_4 \in C([0,+\infty); L^2(0,L)) \cap L^2_{\text{loc}}([0,+\infty); H^1(0,L))$ is the unique solution of

$$\begin{cases} y_{4,t}(t,x) + y_{4,x}(t,x) + y_{4,xxx}(t,x) = 0 & \text{in } (2T_*/3,+\infty)\times(0,L), \\ y_4(t,x=0) = y_4(t,x=L) = 0 & \text{in } (2T_*/3,+\infty), \\ y_{4,x}(t,x=L) = u_4(t) & \text{in } (2T_*/3,+\infty), \\ y_4(T_*/2,\cdot) = y_3(2T_*/3,\cdot). \end{cases}$$

Such a u_4 exists since $y_3(2T_*/3,\cdot)$ is generated from zero at time 0 (see [38]).

Since $y_2(t,\cdot) + y_3(t,\cdot) = 0$ for $t \ge T_*/2$, we have

$$\|u_4\|_{L^2(0,+\infty)} \le C \|y_2(2T_*/3,\cdot)\|_{L^2(0,L)},$$

which yields

$$\begin{aligned} \|u_4\|_{L^2(0,+\infty)} &\overset{\text{Lemma 4.6}}{\le} C \|(f_1,f_2)\|_{L^1(\mathbb{R}_+\times(0,L))} \\ &\le C \min\{\|y\|^2_{L^2((0,T)\times(0,L))}, \|y\|_{L^2((0,T);H^1(0,L))} \|y\|_{L^2((0,T);H^{-1}(0,L))}\} \\ &\overset{(5.12),(5.14),(5.15)}{\le} C \min\{(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})})^2, \varepsilon_0(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})})\}. \end{aligned} \tag{5.17}$$

Let $\widetilde{y} \in C([0,+\infty); L^2(0,L)) \cap L^2_{\text{loc}}([0,+\infty); H^1(0,L))$ be the unique solution of

$$\begin{cases} \widetilde{y}_t(t,x) + \widetilde{y}_x(t,x) + \widetilde{y}_{xxx}(t,x) = 0 & \text{in } (0,+\infty)\times(0,L), \\ \widetilde{y}(t,x=0) = \widetilde{y}(t,x=L) = 0 & \text{in } (0,+\infty), \\ \widetilde{y}_x(t,x=L) = u(t) + u_4(t) & \text{in } (0,+\infty), \\ \widetilde{y}(0,\cdot) = 0, \end{cases}$$

Then, by the choice of u_4,

$$\widetilde{y}(t,\cdot) = 0 \quad \text{for } t \ge T_*.$$

Multiplying the equation of y by $\Psi(t,x)$, integrating by parts on $[0,L]$, and using (5.6) and (5.7), we have

$$\frac{d}{dt}\int_0^L y(t,x)\Psi(t,x)\,dx - \frac{1}{2}\int_0^L y(t,x)^2\Psi_x(t,x)\,dx = 0. \tag{5.18}$$

Integrating (5.18) from 0 to T and using the fact that $y(T,\cdot)=0$ yields

$$\int_0^L y_0(x)\Psi(0,x)\,dx + \frac{1}{2}\int_0^T\int_0^L y(t,x)^2\Psi_x(t,x)\,dx\,dt = 0. \tag{5.19}$$

It is clear that

$$\begin{aligned}
&\left|\int_0^T\int_0^L y(t,x)^2\Psi_x(t,x)\,dx\,dt - \int_0^{+\infty}\int_0^L \tilde{y}(t,x)^2\Psi_x(t,x)\,dx\,dt\right| \\
&\le \left|\int_0^T\int_0^L y(t,x)^2\Psi_x(t,x)\,dx\,dt - \int_0^T\int_0^L y_1(t,x)^2\Psi_x(t,x)\,dx\,dt\right| \\
&\quad + \left|\int_0^{+\infty}\int_0^L y_1(t,x)^2\Psi_x(t,x)\,dx\,dt - \int_0^{+\infty}\int_0^L \tilde{y}(t,x)^2\Psi_x(t,x)\,dx\,dt\right|.
\end{aligned} \tag{5.20}$$

We next estimate the two terms of the RHS of (5.20).

We begin with the first term. We have

$$\begin{aligned}
&\left|\int_0^T\int_0^L y(t,x)^2\Psi_x(t,x)\,dx\,dt - \int_0^T\int_0^L y_1(t,x)^2\Psi_x(t,x)\,dx\,dt\right| \\
&\qquad\qquad \le C\|y-y_1\|_{L^2((0,T);H^1(0,L))}\|(y,y_1)\|_{L^2((0,T);H^{-1}(0,L))}.
\end{aligned} \tag{5.21}$$

By considering the system of $y - y_1$, we obtain

$$\begin{aligned}
\|y-y_1\|_{L^2((0,T);H^1(0,L))} &\le C(\|y_0\|_{L^2(0,L)} + \|f_1\|_{L^1((0,T);L^2(0,L))}) \\
&\le C\|y_0\|_{L^2(0,L)} + C\|y\|^2_{L^2((0,T)\times(0,L))} \\
&\overset{(5.14)}{\le} C\|y_0\|_{L^2(0,L)} + C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})})^2.
\end{aligned} \tag{5.22}$$

Combining (5.15), (5.16), and (5.22), we deduce from (5.21) that

$$\begin{aligned}
&\left|\int_0^T\int_0^L y(t,x)^2\Psi_x(t,x)\,dx\,dt - \int_0^T\int_0^L y_1(t,x)^2\Psi_x(t,x)\,dx\,dt\right| \\
&\le C\varepsilon_0\|y_0\|_{L^2(0,L)} + C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})})(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})})^2.
\end{aligned} \tag{5.23}$$

We next estimate the second term of the RHS of (5.20). It is clear that

$$\begin{aligned}
&\left|\int_0^{+\infty}\int_0^L y_1(t,x)^2\Psi_x(t,x)\,dx\,dt - \int_0^{+\infty}\int_0^L \tilde{y}(t,x)^2\Psi_x(t,x)\,dx\,dt\right| \\
&\le C\|y_1-\tilde{y}\|_{L^2((0,T_*);H^1(0,L))}(\|y_1\|_{L^2((0,T_*);H^{-1}(0,L))} + \|\tilde{y}\|_{L^2((0,T_*);H^{-1}(0,L))}).
\end{aligned} \tag{5.24}$$

Consider the systems of $y_1 - y$ and $\widetilde{y}$. We have

$$\begin{aligned}\|y_1 - \widetilde{y}\|_{L^2((0,T_*);H^1(0,L))} &\le C(\|f\|_{L^1((0,T);L^2(0,L))} + \|u_4\|_{L^2(0,T)})\\ &\overset{(5.17)}{\le} C\|yy_x\|_{L^1((0,T);L^2(0,L))} + C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})})^2\\ &\overset{(5.12)}{\le} C(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)})^2,\end{aligned} \tag{5.25}$$

and, by Lemma 4.6 and (5.17),

$$\|\widetilde{y}\|_{L^2((0,T_*);H^{-1}(0,L))} \le C\|(u, u_4)\|_{H^{-2/3}(\mathbb{R})} \le C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}). \tag{5.26}$$

Using (5.16), (5.25), and (5.26), we deduce from (5.24) that

$$\begin{aligned}\left|\int_0^{+\infty}\int_0^L y_1(t,x)^2\Psi_x(t,x)\,dx\,dt - \int_0^{+\infty}\int_0^L \widetilde{y}(t,x)^2\Psi_x(t,x)\,dx\,dt\right|\\ \le C(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)})^2(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})}).\end{aligned} \tag{5.27}$$

Combining (5.20), (5.23), and (5.27) yields

$$\begin{aligned}&\left|\int_0^{T}\int_0^L y(t,x)^2\Psi_x(t,x)\,dx\,dt - \int_0^{+\infty}\int_0^L \widetilde{y}(t,x)^2\Psi_x(t,x)\,dx\,dt\right|\\ &\le C\varepsilon_0\|y_0\|_{L^2(0,L)} + C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})})(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)})^2.\end{aligned} \tag{5.28}$$

On the other hand, from Corollary 3.7 and the choice of y_0, we have

$$\begin{aligned}\int_0^L y_0(x)\Psi(0,x)\,dx + \frac{1}{2}\int_0^{+\infty}\int_0^L \widetilde{y}(t,x)^2\Psi_x(t,x)\,dx\,dt\\ \ge C(\|y_0\|_{L^2(0,L)} + \|u + u_4\|^2_{H^{-2/3}(\mathbb{R})}).\end{aligned} \tag{5.29}$$

Using the fact that

$$\begin{aligned}\|u + u_4\|^2_{H^{-2/3}(\mathbb{R})} &\ge C\|u\|^2_{H^{-2/3}(\mathbb{R})} - C\|u_4\|^2_{L^2(\mathbb{R})}\\ &\overset{(5.17)}{\ge} C\|u\|^2_{H^{-2/3}(\mathbb{R})} - C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R})})^4,\end{aligned}$$

we infer from (5.29) that, for small ε_0,

$$\begin{aligned}\int_0^L y_0(x)\Psi(0,x)\,dx + \frac{1}{2}\int_0^{\infty}\int_0^L \widetilde{y}(t,x)^2\Psi_x(t,x)\,dx\,dt\\ \ge C(\|y_0\|_{L^2(0,L)} + \|u\|^2_{H^{-2/3}(\mathbb{R})}) - C\|u\|^4_{H^{-1/3}(\mathbb{R})}.\end{aligned} \tag{5.30}$$

Combining (5.19), (5.28), and (5.30) yields

$$
\begin{aligned}
&C\varepsilon_0\|y_0\|_{L^2(0,L)} + C(\|y_0\|_{L^2(0,L)} + \|u\|_{H^{-2/3}(\mathbb{R})})(\|y_0\|_{L^2(0,L)} + \|u\|_{L^2(\mathbb{R}_+)})^2 \\
&\quad \overset{(5.28)}{\geq} \left| \int_0^T \int_0^L y(t,x)^2 \Psi_x(t,x)\,dx\,dt - \int_0^{+\infty} \int_0^L \widetilde{y}(t,x)^2 \Psi_x(t,x)\,dx\,dt \right| \\
&\quad \overset{(5.19)}{\geq} \int_0^L y_0(x)\Psi(0,x)\,dx + \frac{1}{2}\int_0^\infty \int_0^L \widetilde{y}(t,x)^2 \Psi_x(t,x)\,dx\,dt \\
&\quad \overset{(5.30)}{\geq} C(\|y_0\|_{L^2(0,L)} + \|u\|^2_{H^{-2/3}(\mathbb{R})} - C\|u\|^4_{H^{-1/3}(\mathbb{R})}).
\end{aligned} \tag{5.31}
$$

It follows that, if ε_0 is fixed but sufficiently small,

$$
\|u\|^4_{H^{-1/3}(\mathbb{R})} + \|u\|_{H^{-2/3}(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R}_+)} \geq C\|u\|^2_{H^{-2/3}(\mathbb{R})}. \tag{5.32}
$$

We have

$$
\|u\|^2_{H^{-1/3}(\mathbb{R})} \leq C\|u\|_{L^2(\mathbb{R})}\|u\|_{H^{-2/3}(\mathbb{R})} \leq C\varepsilon_0\|u\|_{H^{-2/3}(\mathbb{R})}, \tag{5.33}
$$

$$
\|u\|^2_{L^2(\mathbb{R})} \leq C\|u\|_{H^{-2/3}(\mathbb{R})}\|u\|_{H^{2/3}(\mathbb{R})} \tag{5.34}
$$

(recall that we extended u by 0 for $t < 0$). Let U be the even extension of $u|_{\mathbb{R}_+}$ over $\mathbb{R}$. Applying to U the Hardy inequality for the fractional Sobolev–Slobodetskiĭ space $H^{2/3}(\mathbb{R})$ after noting that $U(0) = 0$ (see e.g. [35, Theorem 1.1])[9], we derive

$$
\big\| |\cdot|^{-2/3} U(\cdot) \big\|_{L^2(\mathbb{R})} \leq C\|U\|_{H^{2/3}(\mathbb{R})}.
$$

We have

$$
\|U\|_{H^{2/3}(\mathbb{R})} \leq C\|u\|_{H^{2/3}(\mathbb{R}_+)}
$$

since U is an even extension of u, and

$$
|U|^2_{H^{2/3}(\mathbb{R})} \sim \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|U(s)-U(t)|^2}{|s-t|^{1+4/3}}\,ds\,dt,
$$

$$
|u|^2_{H^{2/3}(\mathbb{R})} \sim \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{|u(s)-u(t)|^2}{|s-t|^{1+4/3}}\,ds\,dt.
$$

We obtain

$$
\big\| |\cdot|^{-2/3} u(\cdot) \big\|_{L^2(\mathbb{R})} \leq C\|u\|_{H^{2/3}(\mathbb{R}_+)}.
$$

Since

$$
\begin{aligned}
|u|^2_{H^{2/3}(\mathbb{R})} &\sim \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(s)-u(t)|^2}{|s-t|^{1+4/3}}\,ds\,dt \\
&\overset{u(s)=0,\, s<0}{\leq} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{|u(s)-u(t)|^2}{|s-t|^{1+4/3}}\,dx\,dy + C\int_{\mathbb{R}_+} \frac{|u(t)|^2}{t^{4/3}}\,dt \\
&\leq C\|u\|^2_{H^{2/3}(\mathbb{R}_+)} + C\int_{\mathbb{R}_+} \frac{|u(t)|^2}{t^{4/3}}\,dt,
\end{aligned}
$$

[9] We here apply [35, Theorem 1.1 (ii)] with $\gamma = -2/3$, $\tau = p = 2$, $s = 2/3$, $a = 1$, $\alpha = 0$.

it follows that

$$\|u\|_{H^{2/3}(\mathbb{R})} \le C\|u\|_{H^{2/3}(\mathbb{R}_+)}. \tag{5.35}$$

Here we have also used the fact that $u = 0$ in $\mathbb{R}_-$. Combining (5.34) and (5.35) yields

$$\|u\|^2_{L^2(\mathbb{R})} \le C\varepsilon_0\|u\|_{H^{-2/3}(\mathbb{R})}. \tag{5.36}$$

Using (5.33) and (5.36), we deduce from (5.32) that $\|u\|^2_{H^{-2/3}} \le C\varepsilon_0^2\|u\|^2_{H^{-2/3}} + C\varepsilon_0\|u\|^2_{H^{-2/3}}$. So, for fixed sufficiently small ε_0,

$$u = 0.$$

As a consequence, we obtain, by considering the system of $u - \varepsilon\Psi$,

$$\|y(T,\cdot) - \varepsilon\Psi(T,\cdot)\|_{L^2(0,L)} \le C\varepsilon^2.$$

One has a contradiction if ε_0 is sufficiently small. The proof is complete. ∎

Remark 5.5. Viewing the proof of Theorem 5.1, it is natural to ask whether or not one needs to derive estimates for (linear and nonlinear) KdV systems using low regular data. In fact, without using these estimates, one might require that $\|u\|_{H^2(0,T)}$ or even $\|u\|_{H^3(0,T)}$ is small.

6. Controllability of the KdV system with controls in H^1

For $T > 0$, set

$$X = C([0,T];Y) \cap L^2((0,T);H^4([0,L]))$$

with the corresponding norm. Here we denote

$$Y = H^3(0,L) \cap H^1_0(0,L),$$

which is a Hilbert space with the corresponding scalar product.

In this section, we prove the following local controllability result for the KdV system (1.1)–(1.2):

Theorem 6.1. *Let $L > 0$ and $k, l \in \mathbb{N}_*$. Let p be defined by (5.1). Assume that (6.2) holds, $2k + l \notin 3\mathbb{N}_*$, and the dimension of $\mathcal{M}$ is 2. Given $T > \pi/p$, there exists $\varepsilon_0 > 0$ such that for $y_0, y_T \in Y$ with*

$$\|(y_0, y_T)\|_Y \le \varepsilon_0,$$

there exists $u \in H^1(0,T)$ such that $u(0) = y_0'(L)$,

$$\|u\|_{H^1(0,T)} \le C\|(y_0, y_T)\|_Y^{1/2},$$

and the corresponding solution $y \in X$ of the nonlinear system (1.1) with $y(t = 0,\cdot) = y_0$ satisfies $y(t = T,\cdot) = y_T$.

We recall a result of [12, Lemma 3.3] (applied to $s = 3$) on the well-posedness and the stability of the linearized system of (1.1).

Lemma 6.2. *Let $L, T > 0$. For $y_0 \in H^3(0,L) \cap H_0^1(0,L)$, $f \in W^{1,1}([0,T]; L^2(0,L))$, and $u \in H^1(0,T)$ with $u(0) = y_0'(L)$ there exists a unique solution $y \in X$ of the system*

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = f(t,x) & \text{for } t \in (0,T),\ x \in (0,L), \\ y(t,x=0) = y(t,x=L) = 0 & \text{for } t \in (0,T), \\ y_x(t,x=L) = u(t) & \text{for } t \in (0,T), \\ y(t=0,\cdot) = y_0 & \text{for } x \in (0,L). \end{cases} \tag{6.1}$$

Moreover,

$$\|y\|_X \le C(\|f\|_{W^{1,1}([0,T];L^2(0,L))} + \|u\|_{H^1(0,1)})$$

for some positive constant C depending only on L and T.

Remark 6.3. By the same method, the conclusion also holds for nonlinear KdV equations if $\|f\|_{W^{1,1}((0,T);L^2(0,L))} + \|u_0\|_{H^1(0,L)}$ is small.

In the remainder of this section, $\mathcal{M}^\perp$ denotes all elements of Y orthogonal to $\mathcal{M}$ with respect to the $L^2(0,L)$-scalar product. We also denote by $P_{\mathcal{M}}$ and $P_{\mathcal{M}^\perp}$ the projections into $\mathcal{M}$ and $\mathcal{M}^\perp$ with respect to $L^2(0,L)$-scalar product.

For the convenience of the reader, we recall the definition of $\mathcal{M}$. For each $L \in \mathcal{N}$, there exist exactly $n_L \in \mathbb{N}_*$ pairs $(k_m, l_m) \in \mathbb{N}_* \times \mathbb{N}_*$ ($1 \le m \le n_L$) such that $k_m \ge l_m$ and

$$L = 2\pi\sqrt{\frac{k_m^2 + k_m l_m + l_m^2}{3}}. \tag{6.2}$$

For $1 \le m \le n_L$, set

$$p_m = p(k_m, l_m) = \frac{(2k_m + l_m)(k_m - l_m)(2l_m + k_m)}{3\sqrt{3}\,(k_m^2 + k_m l_m + l_m^2)^{3/2}}, \tag{6.3}$$

and denote

$$\begin{cases} \eta_{1,m} = -\dfrac{2\pi i(2k_m + l_m)}{3L}, \\ \eta_{2,m} = \eta_{1,m} + \dfrac{2\pi i}{L}k_m = \dfrac{2\pi i(k_m - l_m)}{3L}, \\ \eta_{3,m} = \eta_{2,m} + \dfrac{2\pi i}{L}l_m = \dfrac{2\pi i(k_m + 2l_m)}{3L}. \end{cases} \tag{6.4}$$

Define, with the convention $\eta_{j+3,m} = \eta_{j,m}$ for $j = 1, 2, 3$,

$$\psi_m(x) = \sum_{j=1}^{3}(\eta_{j+1,m} - \eta_{j,m})e^{\eta_{j+2,m}x} \qquad \text{for } x \in [0,L]. \tag{6.5}$$

Then

$$\mathcal{M} = \operatorname{span}\big\{\{\Re(\psi_m(x));\ 1 \le m \le n_L\} \cup \{\Im(\psi_m(x));\ 1 \le m \le n_L\}\big\}. \tag{6.6}$$

It is clear from the definition of $\eta_{j,m}$ in (6.4) that

$$e^{\eta_{1,m}L} = e^{\eta_{2,m}L} = e^{\eta_{3,m}L}. \tag{6.7}$$

This implies that the function

$$\Psi_m(t,x) = e^{-itp_m}\psi_m(x) \qquad \text{for } (t,x) \in \mathbb{R} \times [0,L]$$

is a solution of the linearized KdV equation which satisfies

$$\Psi_m(t,0) = \Psi_m(t,L) = \partial_x \Psi_m(t,0) = \partial_x \Psi_m(t,L) \tag{6.8}$$

(see also Remark 5.2). In fact, every solution of the linearized KdV equation satisfying the boundary condition given in (6.8) is a linear combination of Ψ_m's for $1 \le m \le n_L$.

Before giving the proof of Theorem 6.1, let us establish two lemmas used in its proof. The first one is a consequence of the Hilbert Uniqueness Method for controls in H^1 and solutions in X.

Lemma 6.4. *Let $L \in \mathcal{N}$ and $T > 0$. There is a continuous linear map $\mathcal{L} : \mathcal{M}^\perp \to H^1(0,T)$ such that if $\varphi \in \mathcal{M}^\perp$ and $u = \mathcal{L}(\varphi)$, then $u(0) = 0$, and the unique solution $y \in X$ of*

$$\begin{cases} y_t(t,x) + y_x(t,x) + y_{xxx}(t,x) = 0 & \textit{for } t \in (0,T),\ x \in (0,L), \\ y(t, x=0) = y(t, x=L) = 0 & \textit{for } t \in (0,T), \\ y_x(t, x=L) = u(t) & \textit{for } t \in (0,T), \\ y(t=0, \cdot) = 0, \end{cases} \tag{6.9}$$

satisfies $y(T,\cdot) = \varphi$.

Proof. Set

$$\mathcal{M}_1^\perp = \{w \in \mathcal{M}^\perp;\ w_x(0) = 0\}.$$

For $\psi \in \mathcal{M}_1^\perp$, by Lemma 6.2, there exists a unique solution $y^* \in X$ of the backward KdV system

$$\begin{cases} y^*_t(t,x) + y^*_x(t,x) + y^*_{xxx}(t,x) = 0 & \text{for } t \in (0,T),\ x \in (0,L), \\ y^*(t, x=0) = y^*(t, x=L) = 0 & \text{for } t \in (0,T), \\ y^*_x(t, x=0) = 0 & \text{for } t \in (0,T), \\ y^*(T, \cdot) = \psi. \end{cases} \tag{6.10}$$

Applying the observability inequality to y^* and y^*_t (see e.g. [18, Theorem 2.4] and also [38, proof of Proposition 3.9]), we have, for $\gamma \ge 1$,

$$\int_{T/2}^T (\gamma |y^*_x(t,L)\eta|^2 + |y^*_{tx}(t,L)|^2)\, dt \ge C \int_0^L (\gamma |y^*(T,x)|^2 + |y^*_t(T,x)|^2)\, dx,$$

where in the last inequality, we use the fact that if $\psi \in \mathcal{M}^\perp$ then $\psi''' + \psi'$ is also in $\mathcal{M}^\perp$ (this can be proved through integration by part arguments; recall that $\mathcal{M}^\perp$ is defined via the $L^2(0, L)$-scalar product). In other words,

$$\int_{T/2}^{T} (\gamma |y_x^*(t, L)|^2 + |y_{tx}^*(t, L)|^2)\, dt \geq C \int_0^L (\gamma |\psi|^2 + |\psi''' + \psi'|^2)\, dx. \tag{6.11}$$

Fix a nonnegative function $\eta \in C^1([0, T])$ such that $\eta = 1$ in $[T/2, T]$ and $\eta = 0$ in $[0, T/3]$. Since

$$\int_0^L (\gamma |\psi|^2 + |\psi''' + \psi'|^2)\, dx = \int_0^L (\gamma |\psi|^2 + |\psi'''|^2 + |\psi'|^2 + 2\psi'''\psi')\, dx,$$

and, for all $\varepsilon > 0$,

$$\int_0^L |\psi'|^2\, dx \leq \int_0^L (\varepsilon |\psi'''|^2 + C_\varepsilon |\psi|^2)\, dx,$$

it follows that, for large γ,

$$\int_0^L (\gamma |\psi|^2 + |\psi''' + \psi'|^2)\, dx \geq C \|\psi\|^2_{H^3(0,L)}. \tag{6.12}$$

We have

$$\begin{aligned}\int_0^T |y_x^*(t, L) y_{tx}^*(t, L)|\, dt &\leq \int_0^T (\varepsilon^{-1} |y_x^*|^2 + \varepsilon |y_{tx}^*|^2)\, dt \\ &\leq C \int_0^L (\varepsilon^{-1} |\psi|^2 + \varepsilon |\psi''' + \psi'|^2)\, dx.\end{aligned}$$

In the last inequalitiy, we have applied [38, (58) in the proof of Proposition 3.7] (see also [18, Proposition 2]) to y^* and y_t^*. It follows from (6.11) and (6.12), for γ large enough, that

$$\int_0^T [\gamma \eta(t) |y_x^*(t, L)|^2 + y_{tx}^*(t, L)(\eta y_x^*(t, L))_t]\, dt \geq C_\gamma \|\psi\|^2_{H^3(0,L)}. \tag{6.13}$$

For a given $\varphi \in \mathcal{M}_1^\perp$, by the Lax–Milgram theorem and (6.13), there exists a unique $\Phi \in \mathcal{M}_1^\perp$ such that

$$\int_0^L [\gamma \varphi \psi + (\varphi''' + \varphi')(\psi''' + \psi')]\, dx = \int_0^T (\gamma y_x^* \eta Y_x^* + y_{tx}^* (\eta Y_x^*)_t)\, dt \quad \forall \psi \in \mathcal{M}_1^\perp, \tag{6.14}$$

where Y^* is the solution of (6.10) with $\psi = \Phi$.

Let $y \in X$ be the solution of (6.9) with $u(\cdot) = \mathcal{L}_1(\varphi) = \eta(\cdot) Y_x^*(\cdot, L)$. Then, by integration by parts,

$$\begin{aligned}&\int_0^L [\gamma \psi y(T, \cdot) + (\psi''' + \psi')(y_{xxx}(T, \cdot) + y_x(T, \cdot))]\, dx \\ &\qquad = \int_0^T (\gamma y_x^* \eta Y_x^* + y_{tx}^* (\eta Y_x^*)_t)\, dt \quad \forall \psi \in \mathcal{M}_1^\perp.\end{aligned} \tag{6.15}$$

From (6.14) and (6.15), we obtain

$$\int_0^L [\gamma\varphi\psi + (\varphi''' + \varphi')(\psi''' + \psi')]$$
$$= \int_0^L [\gamma\psi y(T,\cdot) + (\psi''' + \psi')(y_{xxx}(T,\cdot) + y_x(T,\cdot))] \quad \forall \psi \in \mathcal{M}_1^\perp.$$

Since y and Y^* satisfies system (6.9) with the same u for $t \in [T/2, T]$, it follows that $y(t,\cdot) - Y^*(t,\cdot) \in \mathcal{M}$ for $t \in [T/2, T]$. In particular, $y(T,\cdot) \in \mathcal{M}_1^\perp$ since $Y^*(T,\cdot) \in \mathcal{M}_1^\perp$. Combining this with the fact that $\varphi \in \mathcal{M}_1^\perp$, we deduce from (6.12) that

$$y(T,\cdot) = \varphi.$$

The conclusion for $2T$ (instead of T) is now as follows. Fix $\zeta \in C^1([0,2T])$ with $\zeta(2T) = 1$ and $\zeta(t) = 0$ for $t \le 5T/4$. For $\varphi \in \mathcal{M}^\perp$, let $\widetilde{y}^*$ be the unique solution of

$$\begin{cases} \widetilde{y}^*_t(t,x) + \widetilde{y}^*_x(t,x) + \widetilde{y}^*_{xxx}(t,x) = 0 & \text{for } t \in (T,2T),\ x \in (0,L), \\ \widetilde{y}^*(t,x=0) = \widetilde{y}^*(t,x=L) = 0 & \text{for } t \in (T,2T), \\ \widetilde{y}^*_x(t,x=0) = \varphi_x(2T,0)\zeta(t) & \text{for } t \in (T,2T), \\ \widetilde{y}^*(2T,\cdot) = \varphi. & \end{cases}$$

One can check that $\widetilde{y}^*(T,\cdot) \in \mathcal{M}_1^\perp$. Set

$$\mathcal{L}(\varphi)(t) = \begin{cases} \widetilde{y}^*_x(t,L) & \text{for } t \in (T,2T), \\ \mathcal{L}_1(\widetilde{y}^*(T,\cdot))(t) & \text{for } t \in (0,T). \end{cases} \tag{6.16}$$

It is clear that $\mathcal{L}(\varphi) \in H^1(0,2T)$ since $\widetilde{y}_x(\cdot,L) \in H^1(T,2T)$, $\mathcal{L}_1(\widetilde{y}^*(T,\cdot)) \in H^1(0,T)$, and $\mathcal{L}_1(\widetilde{y}^*(T,\cdot))(T) = \widetilde{y}^*_x(T,L)$, and that the corresponding solution at time $2T$ is φ. The proof is complete. ■

For $r > 0$ and an element $e \in Y$, we denote by $B_r(e)$ the open ball in Y centered at e with radius r, and open $\overline{B_r(e)}$ its closure in Y. The second lemma is a consequence of the power series method and the information derived in Sections 3 and 5.

Lemma 6.5. *Let $L > 0$ and $k, l \in \mathbb{N}_*$. Let p be defined by (5.1). Assume that (6.2) holds, $2k + l \notin 3\mathbb{N}_*$, and the dimension of $\mathcal{M}$ is 2. Let $T > \pi/p$ and $0 < c_1 < c_2$. Fix $\varphi \in \mathcal{M}$ with $c_1 \le \|\varphi\|_Y \le c_2$. There exist a constant $0 < c_3 < c_1/2$, and two maps $U_1 : B_{c_3}(\varphi) \to H^1(0,T)$ and $U_2 : B_{c_3}(\varphi) \to H^1(0,T)$ such that $U_1(\varphi)(0) = U_2(\varphi)(0) = 0$, and for $\psi \in B_{c_3}(\varphi)$, the unique solutions y_1 and y_2 in X of the following two systems, with $u_1 = U_1(\varphi)$ and $u_2 = U_2(\varphi)$:*

$$\begin{cases} y_{1,t}(t,x) + y_{1,x}(t,x) + y_{1,xxx}(t,x) = 0 & \textit{for } t \in (0,T),\ x \in (0,L), \\ y_1(t,x=0) = y_1(t,x=L) = 0 & \textit{for } t \in (0,T), \\ y_{1,x}(t,x=L) = u_1(t) & \textit{for } t \in (0,T), \\ y_1(t=0,\cdot) = 0 & \textit{for } t \in (0,T), \end{cases} \tag{6.17}$$

$$\begin{cases} y_{2,t}(t,x) + y_{2,x}(t,x) + y_{2,xxx}(t,x) + y_1(t,x)y_{1,x}(t,x) = 0 & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } t \in (0,T),\ x \in (0,L), & \\ y_2(t, x = 0) = y_2(t, x = L) = 0 & \textit{for } t \in (0,T), \\ y_{2,x}(t, x = L) = u_2(t) & \textit{for } t \in (0,T), \\ y_1(t = 0, \cdot) = 0 & \textit{for } t \in (0,T), \end{cases} \tag{6.18}$$

satisfy

$$y_1(T,\cdot) = 0 \qquad \textit{and} \qquad y_2(T,\cdot) = \psi.$$

Moreover, for $\psi, \widetilde{\psi} \in B_{c_3}(\varphi)$,

$$\|U_1(\psi) - U_1(\widetilde{\psi})\|_{H^1(0,T)} \le C\|\psi - \widetilde{\psi}\|_Y \tag{6.19}$$

and

$$\|U_2(\psi) - U_2(\widetilde{\psi})\|_{H^1(0,T)} \le C\|\psi - \widetilde{\psi}\|_Y, \tag{6.20}$$

for some positive constant C *depending only on* L, T, c_1, *and* c_2.

Proof. By Lemma 5.3 and Corollary 3.7, for all $\tau > 0$, there exists $v_1 \in H^2_0(0,\tau)$ such that if $y_1 \in X$ is the solution of (6.17) with $u_1 = v_1$ and $y_2 \in X$ is the solution of (6.18) with $u_2 = 0$ then

$$y_2(\tau,\cdot) \in \mathcal{M} \setminus \{0\}.$$

Since c_3 is small and $\dim \mathcal{M} = 2$, and $v_1 \in H^2_0(0,L)$, by using rotations (see also [18, proof of Proposition 13]) there exists $U_1(\psi)$ with $U_1(\psi)(0) = 0$ satisfying (6.19) such that if $y_1 \in X$ is the solution of (6.17) with $u_1 = U_1(\psi)$ and $\hat{y}_2 \in X$ is the solution of (6.18) with $u_2 = 0$ then

$$\hat{y}_2 = P_{\mathcal{M}}\psi.$$

We then choose

$$u_2 = \mathcal{L}(\hat{y}_2 - P_{\mathcal{M}}\psi),$$

where $\mathcal{L}$ is the map given by Lemma 6.4. ■

Proof of Theorem 6.1. Fix $y_0, y_T \in Y$ with small norms. For simplicity of presentation, we will assume that $\|y_0\|_Y \le \|y_T\|_Y$ (the other case also follows from this case by e.g. reversing the time: $t \to T - t$ and noting that $y_x(\cdot,0)$ is in $H^1(0,T)$; this can be derived by considering the equation for y_t [10]). Set $\rho = \|y_T\|_Y$ and assume that $\rho > 0$; otherwise, one just takes the zero control and the conclusion follows.

Let w_0 be the state at time T of the solution of the linear system (6.9) with the zero control starting from $P_{\mathcal{M}}y_0$ at time 0. We first consider the case where

$$\|P_{\mathcal{M}}y_T - w_0\|_{H^2(0,L)} \ge 2c\rho \tag{6.21}$$

for some small constant c independent of ρ and to be defined later.

[10] The compatibility condition is automatic.

Set

$$\mathbb{G}: Y \cap \overline{B_{c\rho}(y_T)} \to H^1(0,T), \quad \varphi \mapsto \rho \mathbf{u}_0 + \rho^{1/2} u_1 + \rho u_2.$$

Here we decompose φ as

$$\varphi = P_{\mathcal{M}^\perp}\varphi + P_{\mathcal{M}}\varphi,$$

$\mathbf{u}_0 \in H^1(0,T)$ is a control for which the corresponding solution $\mathbf{y}_0$ in X of the linear system (6.9) starting from $P_{\mathcal{M}^\perp} y_0/\rho$ at 0 and arriving at $P_{\mathcal{M}^\perp}\varphi/\rho$ at time T, and u_1 and u_2 are controls for which the solutions $y_1 \in X$ and $y_2 \in X$ of the system (6.17)–(6.18) (with initial data $P_{\mathcal{M}} y_0/\rho$ instead of 0) satisfy $y_1(T,\cdot) = 0$ and $y_2(T,\cdot) = P_{\mathcal{M}}\varphi/\rho$. Moreover, by Lemma 6.4, one can choose $\mathbf{u}_0$ in such a way that $\mathbf{u}_0 = \mathbf{u}_0(\varphi)$ is a Lipschitz function of φ with Lipschitz constant bounded by a positive constant independent of ρ, and by Lemma 6.5 one can choose $u_1 = u_1(\varphi)$ and $u_2 = u_2(\varphi)$ as Lipschitz functions of $P_{\mathcal{M}}\varphi/\rho$ with Lipschitz constants bounded by positive constants independent of ρ.

Set

$$\mathbb{P}: \{w \in H^1(0,T);\, w(0) = y_0'(L)\} \to H^3(0,L), \quad w \mapsto y(T,\cdot),$$

where $y \in X$ is the unique solution of the nonlinear system (1.1) with $u = w$ starting from y_0 at time 0. Consider the map

$$\Lambda: Y \cap \overline{B_{c\rho}(y_T)} \to Y, \quad \varphi \mapsto \varphi - \mathbb{P} \circ \mathbb{G}(\varphi) + y_T.$$

We will prove that

$$\Lambda(\varphi) \in \overline{B_{c\rho}(y_T)}, \tag{6.22}$$

and

$$\|\Lambda(\varphi) - \Lambda(\phi)\|_Y \le \lambda \|\varphi - \phi\|_Y \tag{6.23}$$

for some $\lambda \in (0,1)$. Assuming this, one infers from the contraction mapping theorem that there exists a unique $\varphi_0 \in Y \cap \overline{B_{c\rho}(y_T)}$ such that $\Lambda(\varphi_0) = \varphi_0$. As a consequence,

$$y_T = \mathbb{P} \circ \mathbb{G}(\varphi_0),$$

and $\mathbb{G}(\varphi_0)$ is hence a required control.

We next establish (6.22) and (6.23). Indeed, (6.22) follows from the inequality

$$\|\varphi - \mathbb{P} \circ \mathbb{G}(\varphi)\|_Y \le C \|\varphi\|_Y^{3/2} \quad \text{for } \varphi \in Y \cap \overline{B_{c\rho}(y_T)}.$$

This can be proved using approximation via the power series method as follows. Set[11]

$$u = \rho \mathbf{u}_0 + \rho^{1/2} u_1 + \rho u_2 \quad \text{and} \quad y_a = \rho \mathbf{y}_0 + \rho^{1/2} y_1 + \rho y_2.$$

Let $y \in X$ be the solution of the nonlinear KdV system (1.1) with $y(t=0,\cdot) = y_0$ and with u defined above. Then

$$(y - y_a)_t + (y - y_a)_x + (y - y_a)_{xxx} + y y_x - y_a y_{a,x} = f(t,x),$$

[11]The subscript a stands for approximation.

where

$$-f(t,x) = \rho^{3/2}(y_1 y_2)_x + \rho^2 y_2 y_{2,x} + \rho^2 \mathbf{y_0} \mathbf{y}_{0,x} + \rho^{3/2}(\mathbf{y_0}(y_1 + \rho^{1/2} y_2))_x.$$

Since

$$y y_x - y_a y_{a,x} = (y - y_a) y_x + y_a (y_x - y_{a,x}),$$

applying Lemma 6.2 we obtain, for small ρ,

$$\|y - y_a\|_X \le C \|f\|_{W^{1,1}((0,T);L^2(0,L))} \le C\rho^{3/2}. \tag{6.24}$$

Assertion (6.22) follows since $y(T,\cdot) = \mathbb{P} \circ \mathbb{G}(\varphi)$ and $y_a(T,\cdot) = \varphi$.

We next establish (6.23). To this end, we estimate

$$(\varphi - \mathbb{P} \circ \mathbb{G}(\varphi)) - (\widetilde{\varphi} - \mathbb{P} \circ \mathbb{G}(\widetilde{\varphi})).$$

Denote by $\widetilde{\mathbf{u}}_0, \widetilde{u}_1, \widetilde{u}_2, \widetilde{u}$ and $\widetilde{\mathbf{y}}_0, \widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_a, \widetilde{y}$ the functions corresponding to $\widetilde{\varphi}$ which are defined in the same way as the functions $\mathbf{u}_0, u_1, u_2, u$ and $\mathbf{y}_0, y_1, y_2, y_a, y$ defined for φ.

We have

$$\begin{aligned}
&(y - \widetilde{y})_t + (y - \widetilde{y})_x + (y - \widetilde{y})_{xxx} + y y_x - \widetilde{y}\widetilde{y}_x = 0,\\
&(y_a - \widetilde{y}_a)_t + (y_a - \widetilde{y}_a)_x + (y_a - \widetilde{y}_a)_{xxx} + y_a y_{a,x} - \widetilde{y}_a \widetilde{y}_{a,x} = g(t,x),
\end{aligned}$$

where

$$\begin{aligned}
g(t,x) = \rho^{3/2}((y_1 y_2)_x - (\widetilde{y}_1 \widetilde{y}_2)_x) + \rho^2 (y_2 y_{2,x} - \widetilde{y}_2 \widetilde{y}_{2,x}) + \rho^2 (\mathbf{y}_0 \mathbf{y}_{0,x} - \widetilde{\mathbf{y}}_0 \widetilde{\mathbf{y}}_{0,x})\\
+ \rho^{3/2}\big(\mathbf{y}_0(y_1 + \rho^{1/2} y_2) - \widetilde{\mathbf{y}}_0(\widetilde{y}_1 + \rho^{1/2}\widetilde{y}_2)\big)_x.
\end{aligned} \tag{6.25}$$

This implies

$$\begin{aligned}
&(y - y_a - \widetilde{y} + \widetilde{y}_a)_t + (y - y_a - \widetilde{y} + \widetilde{y}_a)_x + (y - y_a - \widetilde{y} + \widetilde{y}_a)_{xxx}\\
&= -((y - y_a) y_x + y_a (y - y_a)_x - (\widetilde{y} - \widetilde{y}_a)\widetilde{y}_x - \widetilde{y}_a(\widetilde{y} - \widetilde{y}_a)_x + g(t,x))\\
&= -\big((y - y_a - \widetilde{y} + \widetilde{y}_a) y_x + (y_x - \widetilde{y}_x)(\widetilde{y} - \widetilde{y}_a) + y_a (y - y_a - \widetilde{y} + \widetilde{y}_a)_x\\
&\qquad\qquad + (y_a - \widetilde{y}_a)(\widetilde{y} - \widetilde{y}_a)_x + g(t,x)\big)\\
&= -\big((y - y_a - \widetilde{y} + \widetilde{y}_a) y_x + y_a (y - y_a - \widetilde{y} + \widetilde{y}_a)_x\\
&\qquad\qquad + (y_x - y_{a,x} - \widetilde{y}_x + \widetilde{y}_{a,x})(\widetilde{y} - \widetilde{y}_a) + h(t,x)\big),
\end{aligned}$$

where

$$h(t,x) = g(t,x) + (y_{a,x} - \widetilde{y}_{a,x})(\widetilde{y} - \widetilde{y}_a) + (y_a - \widetilde{y}_a)(\widetilde{y} - \widetilde{y}_a)_x.$$

Using Lemma 6.2, we find that, for ρ small,

$$\|y - y_a - \widetilde{y} + \widetilde{y}_a\|_X \le C \|h(t,x)\|_{W^{1,1}((0,T);L^2(0,L))}. \tag{6.26}$$

We have

$$\|(y - y_a, \widetilde{y} - \widetilde{y}_a)\|_X \overset{(6.24)}{\le} C\rho^{3/2}, \quad \|y_a - \widetilde{y}_a\|_X \le C\rho^{-1/2}\|\varphi - \widetilde{\varphi}\|_Y,$$

and

$$\|g(t,x)\|_{W^{1,1}((0,T);L^2(0,L))} \leq C\rho^{1/2}\|\varphi - \widetilde{\varphi}\|_Y.$$

It follows that

$$\|h(t,x)\|_{W^{1,1}((0,T);L^2(0,L))} \leq C\rho^{1/2}\|\varphi - \phi\|_Y, \tag{6.27}$$

which yields, by (6.26),

$$\|(y - y_a - \widetilde{y} + \widetilde{y}_a)(T,\cdot)\|_Y \leq C\rho^{1/2}\|\varphi - \phi\|_Y.$$

Assertion (6.23) follows.

We next consider the case $\|P_{\mathcal{M}} y_T - w_0\|_{H^3(0,L)} \leq 2c\|y_T\|_{H^3(0,L)}$. In fact, one can reduce this case to the previous one as follows. Fix $\varepsilon > 0$ small. By Lemma 5.3 and Corollary 3.7, there exists $v_1 \in H_0^2(0,\varepsilon)$ such that if $y_1 \in X$ (with $T = \varepsilon$) is the solution of (6.17) with $u_1 = v_1$ and $y_2 \in X$ is the solution of (6.18) with $u_2 = 0$ then

$$y_2(\varepsilon,\cdot) \in \mathcal{M} \setminus \{0\}.$$

Let $u_{0,T}$ be a control for which the corresponding solution in X of the linear system (6.9) starts from $y_T(L-\cdot)/\rho$ at 0 and arrives at 0 at time ε, and set $u_{1,T} = \gamma v_1$, $u_{2,T} = \gamma^2 v_2$ for some $\gamma > 0$ to be defined later. Let $\mathbf{y}$ be the unique solution of the nonlinear KdV system in the time interval $[T, T+\varepsilon]$ using the control

$$\rho u_0(\cdot - T) + \rho^{1/2} u_1(\cdot - T) + \rho u_2(\cdot - T)$$

with $\mathbf{y}(T,\cdot) = y_T(L-\cdot)$. By choosing γ large enough, y_0 and $\mathbf{y}(T+\varepsilon, L-\cdot)$ satisfy the setting of the previous case for the time interval $[0, T+\varepsilon]$ (instead of $[0,T]$). One now considers the control (for the nonlinear KdV system) in the time interval $[0, T+2\varepsilon]$ which is equal to the one which brings y_0 at time 0 to $\mathbf{y}(T+\varepsilon, L-\cdot)$ at time $T+\varepsilon$ obtained in the previous case in the time interval $[0, T+\varepsilon]$, and is equal to $-\mathbf{y}_x(2(T+\varepsilon)-t, 0)$ for $t \in [T+\varepsilon, T+2\varepsilon]$. It is clear that the solution of the nonlinear KdV system at time $T+2\varepsilon$ is y_T. The proof is completed by changing $T+2\varepsilon$ to T. ∎

Remark 6.6. A similar result to Theorem 6.1 also holds for $y_0, y_T \in H^2(0,L) \cap H_0^1(0,L)$ and $u \in H^{2/3}(0,T)$. More precisely, one has the following result. Let $L > 0$, and $k, l \in \mathbb{N}_*$. Let p be defined by (5.1). Assume that (6.2) holds, $2k + l \not\in 3\mathbb{N}_*$, and the dimension of $\mathcal{M}$ is 2. Given $T > \pi/p$, there exists $\varepsilon_0 > 0$ such that for $y_0, y_T \in H^2(0,L) \cap H_0^1(0,L)$ with

$$\|(y_0, y_T)\|_{H^2(0,L)} \leq \varepsilon_0,$$

there exists $u \in H^{2/3}(0,T)$ such that $u(0) = y_0'(L)$,

$$\|u\|_{H^{2/3}(0,T)} \leq C\|(y_0, y_T)\|_{H^2}^{1/2},$$

and the corresponding solution $y \in C([0,T]; H^2(0,L)) \cap L^2((0,T); H^3[0,L]))$ of the nonlinear system (1.1) with $y(t=0,\cdot) = y_0$ satisfies $y(t=T,\cdot) = y_T$. This is complementary to Theorem 5.1. The only important modification in comparison with the proof of

Theorem 6.1 is Lemma 6.4. Nevertheless, the method presented in its proof can be extended to cover the setting described here (initial and final datum in $H^2(0,L) \cap H_0^1(0,1)$ and controls in $H^{2/3}(0,T)$). We also have

$$\|y_x(\cdot,0)\|_{H^{2/3}(0,T)} \le C\big(\|y(0,\cdot)\|_{H^2(0,L)} + \|y_x(\cdot,L)\|_{H^{2/3}(0,T)}\big) \tag{6.28}$$

for solutions $y \in C([0,T]; H^2(0,L)) \cap L^2((0,T); H^3[0,L]))$ of (1.1) with small norm. Assertion (6.28) would follow from [12] applied to $s = 2$. Here is another way to see it. Split y into two parts y_1 and y_2 where y_1 is the solution of the linearized system with zero initial data and $y_{1,x}(\cdot,L) = y_x(\cdot,L)$. As in the proof of Lemma 4.4, one can prove

$$\|y_{1,x}(\cdot,0)\|_{H^{2/3}(0,T)} \le C\|y_x(\cdot,L)\|_{H^{2/3}(0,T)}. \tag{6.29}$$

Concerning y_2, by considering yy_x as a source term, similar to the proof of Lemma 4.6, one can prove

$$\|y_{2,x}(\cdot,0)\|_{H^{2/3}(0,T)} \le C\big(\|y(0,\cdot)\|_{H^2(0,L)} + \|yy_x\|_{L^2((0,T);H^2(0,L))}\big). \tag{6.30}$$

Since

$$\begin{aligned}\|yy_x\|_{L^2((0,T);H^2(0,L))} &\le C\|y\|^2_{C([0,T];H^2(0,L))\cap L^2((0,T);H^3[0,L]))} \\ &\qquad\qquad \text{(by the embedding theorem)} \\ &\le C\big(\|y(0,\cdot)\|_{H^2(0,L)} + \|y_x(\cdot,L)\|_{H^{2/3}(0,T)}\big)^2 \\ &\qquad\qquad \text{(by [12, Theorem 3.4] applied to } s = 2\text{)},\end{aligned}$$

assertion (6.28) follows from (6.29) and (6.30). Therefore, the arguments using the backward systems also work in this case.

Remark 6.7. The proof given in Theorem 6.1 can be easily extended to the case of $L \notin \mathcal{N}$ to yield the small-time local controllability of (1.1) with initial and final datum in $H^3(0,L) \cap H_0^1(0,L)$ (resp. $H^2(0,L) \cap H_0^1(0,L)$) and controls in $H^1(0,T)$ (resp. $H^{2/3}(0,T)$).

Remark 6.8. Let $L \in \mathcal{N}$. Assume that $\dim \mathcal{M}$ is even and for all $(k,l) \in \mathbb{N}_*^2$ such that $k > l \ge 1$ and $L = \frac{1}{2\pi}\sqrt{\frac{k^2+l^2+kl}{3}}$, one has $2k + l \notin 3\mathbb{N}_*$. Then, using the same method as in the proof of Theorem 6.1, and involving the ideas of [20], one can prove that system (1.1)–(1.2) is controllable in time given in [20].

Remark 6.9. The mappings $\mathbb{G}$ and Λ have their roots in [24] (see also [18]).

Remark 6.10. Lemma 6.4 is motivated by the Hilbert Uniqueness Method and inspired by the construction of smooth controls (for different contexts, e.g. the context of the wave equation) in [27]. The function η used there is inspired by [27]. Nevertheless, we cannot take $\eta = 0$ near T as in [27]. We also add a large parameter λ in the proof.

Remark 6.11. In the proof of Lemma 6.5, we essentially use the fact that for all $\tau > 0$, there exists $v_1 \in H_0^2(0,\tau)$ such that if $y_1 \in X$ is the solution of (6.17) with $u_1 = v_1$ and $y_2 \in X$ is the solution of (6.18) with $u_2 = 0$ then

$$y_2(\tau, \cdot) \in \mathcal{M} \setminus \{0\}.$$

This is a consequence of Lemma 5.3 and Corollary 3.7. It is not clear to us how to use a contradiction argument as in [18, 20, 24] to obtain such a function v_1. This is why we cannot implement the strategy of [18, 20, 24] to derive the local controllability for initial and final datum in $H^3(0,L) \cap H_0^1(0,L)$ with controls in $H^1(0,T)$ for all critical lengths and for small time when $\dim \mathcal{M} = 1$, and in finite time otherwise.

Remark 6.12. We emphasize that the way of implementing the fixed point argument for Λ presented in this paper is somewhat different from the one in [18]. We only apply the fixed point arguments once, instead of twice, first for $P_{\mathcal{M}^\perp}\Lambda$ and then for $P_{\mathcal{M}}\Lambda$, as in [18]. The Brouwer fixed point theorem is not required in our analysis.

Appendix A. On symmetric functions of the roots of a polynomial

This is standard for people knowing algebraic functions [1, Ch. 8, §2], but for the sake of completeness, we justify that an analytic symmetric function of the roots $\lambda_j(z)$ of $\lambda^3 + \lambda + iz = 0$ is an entire function.

Lemma A.1. *Let $(\lambda_1(z), \lambda_2(z), \lambda_3(z))$ be the three roots of $\lambda^3 + \lambda + iz = 0$. Let $F\colon \mathbb{C}^3 \to \mathbb{C}$ be holomorphic in $\mathbb{C}^3$ and symmetric, i.e., for every permutation $\sigma \in \mathfrak{S}_3$, $F(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) = F(z_1, z_2, z_3)$. Then the function $G\colon \mathbb{C} \ni z \mapsto F(\lambda_1(z), \lambda_2(z), \lambda_3(z))$ is entire.*

Note that the ordering $\lambda_1(z), \lambda_2(z), \lambda_3(z)$ is not unique (and we could prove that we cannot chose an ordering that makes any of the λ_j entire), but since F is symmetric, the value $F(\lambda_1(z), \lambda_2(z), \lambda_3(z))$ does not depend on the ordering.

Proof of Lemma A.1. Note that, for $z_0 \neq \pm 2/(3\sqrt{3})$, the discriminant of $X^3 + X + iz$ is nonzero, and thus the roots of $X^3 + X + iz_0$ are simple. By the implicit function theorem, there exists some complex neighborhood U of z_0, some neighborhood V_j of $\lambda_j(z_0)$ $(1 \le j \le 3)$, and three holomorphic functions $\mu_j\colon U \to V_j$ such that $\mu_1(z)$, $\mu_2(z)$, $\mu_3(z)$ are the three distinct roots. Since F is symmetric, it follows that $G(z) = F(\mu_1(z), \mu_2(z), \mu_3(z))$ and is therefore analytic in U. Consequently, G is analytic in $\mathbb{C} \setminus \{\pm 2/(3\sqrt{3})\}$.

It then suffices to prove that G is continuous at $\pm 2/(3\sqrt{3})$. The roots $\lambda_j(z)$ are continuous, even at $\pm\sqrt{4/27}$, in the sense that for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $|z - z_0| < \delta$, there exists some ordering of the $\lambda_{k_j}(z)$ such that $|\lambda_{k_1}(z) - \lambda_1(z_0)| + \cdots + |\lambda_{k_3}(z) - \lambda_3(z_0)| < \epsilon$ (this can be seen e.g. thanks to Cardano's formula). Thus $G(z)$ is continuous at $z_0 = \pm\sqrt{4/27}$. ∎

Remark A.2. A variant of Lemma A.1 still holds for more general polynomial equations $P(z,\lambda)=0$, but we wanted to avoid some technicalities. The general case would be a consequence of the fact that the solutions of $P(z,\lambda)=0$ define a finite number of algebraic functions [1, Ch. 8, §2].

Appendix B. On the real roots of H, the common roots of G and H, and the behavior of $|\det Q|$

Lemma B.1. *Let $z\in\mathbb{R}$.*

(1) *If $z\neq\pm 2/(3\sqrt{3})$ and $H(z)=0$, then, for some $k,l\in\mathbb{N}_*$ with $1\le l\le k$, we have $L=2\pi\sqrt{\frac{k^2+kl+l^2}{3}}$ and*

$$z=-\frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}\,(k^2+kl+l^2)^{3/2}}. \tag{B.1}$$

Moreover,

$$\lambda_1(z)=-\frac{2\pi i}{3L}(2k+l),\quad \lambda_2(z)=\lambda_1(z)+\frac{2\pi i}{L}k,\quad \lambda_3(z)=\lambda_2(z)+\frac{2\pi i}{L}l, \tag{B.2}$$

and z is a simple zero of H.

(2) *If $z=\pm 2/(3\sqrt{3})$ then*

$$\lambda_1(z)=\mp\frac{i}{\sqrt{3}},\quad \lambda_2(z)=\mp\frac{i}{\sqrt{3}},\quad \lambda_3(z)=\pm\frac{2i}{\sqrt{3}}, \tag{B.3}$$

z is not a zero of H, and z is a simple solution of the equation $\det Q(z)\Xi(z)=0$.

Proof. (1) By Remark 2.7, assertion (B.1) holds. Assertion (B.2) then follows from [38]. To prove that z is then a simple root of $H(z)=0$ in the case $z\neq\pm 2/(3\sqrt{3})$, we proceed as follows. We have

$$\lambda_j(z+\varepsilon)=\lambda_j(z)-\frac{i\varepsilon}{3\lambda_j^2+1}+O(\varepsilon^2).$$

It follows that

$$\begin{aligned}\det Q(z+\varepsilon)&=\sum_{j=1}^{3}(\lambda_{j+1}(z+\varepsilon)-\lambda_j(z+\varepsilon))e^{-\lambda_{j+2}(z+\varepsilon)L}\\&=\sum_{j=1}^{3}\Big(\lambda_{j+1}(z)-\lambda_j(z)-\frac{i\varepsilon}{3\lambda_{j+1}^2+1}+\frac{i\varepsilon}{3\lambda_j^2+1}+O(\varepsilon^2)\Big)e^{-\lambda_{j+2}(z)L}\\&\qquad\cdot\Big(1+\frac{i\varepsilon L}{3\lambda_{j+2}^2+1}+O(\varepsilon^2)\Big).\end{aligned}$$

Since

$$e^{-\lambda_1(z)L}=e^{-\lambda_2(z)L}=e^{-\lambda_3(z)L},$$

we derive

$$\det Q(z+\varepsilon) = i\varepsilon L e^{-\lambda_1(z)L} \sum_{j=1}^{3} \frac{\lambda_{j+1}(z)-\lambda_j(z)}{3\lambda_{j+2}^2(z)+1} + O(\varepsilon^2). \tag{B.4}$$

In what follows, for notational ease, we denote $\lambda_j(z)$ by λ_j. We have

$$\begin{aligned}\sum_{j=1}^{3} \frac{\lambda_{j+1}-\lambda_j}{3\lambda_{j+2}^2+1} &= \frac{2\pi i}{L}\left(\frac{k}{3\lambda_3^2+1} + \frac{l}{3\lambda_1^2+1} - \frac{k+l}{3\lambda_2^2+1}\right)\\ &= \frac{2\pi i}{L}\left(\frac{3k(\lambda_2^2-\lambda_3^2)}{(3\lambda_3^2+1)(3\lambda_2^2+1)} + \frac{3l(\lambda_2^2-\lambda_1^2)}{(3\lambda_1^2+1)(3\lambda_2^2+1)}\right)\\ &= \left(\frac{2\pi i}{L}\right)^2\left(-\frac{3kl(\lambda_2+\lambda_3)}{(3\lambda_3^2+1)(3\lambda_2^2+1)} + \frac{3kl(\lambda_2+\lambda_1)}{(3\lambda_1^2+1)(3\lambda_2^2+1)}\right).\end{aligned} \tag{B.5}$$

Note that

$$\begin{aligned}&(\lambda_2+\lambda_1)(3\lambda_3^2+1) - (\lambda_2+\lambda_3)(3\lambda_1^2+1)\\ &\qquad = (\lambda_1-\lambda_3) + 3(\lambda_3-\lambda_1)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) = 2(\lambda_3-\lambda_1),\end{aligned} \tag{B.6}$$

since $\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 = 1$. From (B.4)–(B.6), we deduce that z is a simple root of $H(z)$.

(2) We only consider the case $z = 2/(3\sqrt{3})$; the other case follows similarly. By (2.19) in the proof of Lemma 2.6, we have

$$\begin{aligned}&\lambda_1(z+\varepsilon) = -\frac{i}{\sqrt{3}} + \frac{\sqrt{-i}}{3^{1/4}}\sqrt{\epsilon} + O(\epsilon),\\ &\lambda_2(z+\varepsilon) = -\frac{i}{\sqrt{3}} - \frac{\sqrt{-i}}{3^{1/4}}\sqrt{\epsilon} + O(\epsilon),\ \lambda_3(z+\epsilon) = \frac{2i}{\sqrt{3}} + O(\epsilon).\end{aligned} \tag{B.7}$$

It follows that

$$\det Q(z+\varepsilon) = -\frac{2Li}{\sqrt{3}}\frac{\sqrt{-i}}{3^{1/4}}\sqrt{\varepsilon} + O(\varepsilon).$$

Since $\Xi(z+\varepsilon) = c_+\sqrt{\varepsilon}$ for some $c_+ \neq 0$ by (B.7), $z = 2/(3\sqrt{3})$ is not a root of $H(z) = 0$ and z is a simple root of $\det Q(z)\Xi(z) = 0$. The proof is complete. ■

Lemma B.2. *Let $z \in \mathbb{C}$ be such that $z \neq \pm 2/(3\sqrt{3})$. Assume that $H(z) = G(z) = 0$. Then, for some $k, l \in \mathbb{N}_*$ with $k \geq l \geq 1$, we have*

$$L = 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, \tag{B.8}$$

and

$$z = -\frac{(2k+l)(k-l)(2l+k)}{3\sqrt{3}(k^2+kl+l^2)^{3/2}}. \tag{B.9}$$

Proof. By Remark 2.7 (see also Lemma B.1), it suffices to prove that if $z \in \mathbb{C}$ is such that $z \neq \pm 2/(3\sqrt{3})$, and $H(z) = G(z) = 0$, then z is real. Indeed, note that

$$\det Q(z) = (\lambda_1 - \lambda_3)(e^{-\lambda_2 L} - e^{-\lambda_3 L}) + (\lambda_3 - \lambda_2)(e^{-\lambda_1 L} - e^{-\lambda_3 L}),$$

and

$$-P(z) = (\lambda_1 - \lambda_3)(e^{\lambda_2 L} - e^{\lambda_3 L}) + (\lambda_3 - \lambda_2)(e^{\lambda_1 L} - e^{\lambda_3 L}).$$

It follows that

$$|\det Q(z)| = 0 \quad \text{if and only if} \quad (\lambda_3 - \lambda_1)(e^{(\lambda_3 - \lambda_2)L} - 1) = (\lambda_3 - \lambda_2)(e^{(\lambda_3 - \lambda_1)L} - 1), \tag{B.10}$$

and

$$|P(z)| = 0 \quad \text{if and only if} \quad (\lambda_3 - \lambda_1)(e^{-(\lambda_3 - \lambda_2)L} - 1) = (\lambda_3 - \lambda_2)(e^{-(\lambda_3 - \lambda_1)L} - 1). \tag{B.11}$$

Solving the system

$$\begin{cases} \sum_{j=1}^{3} \lambda_j = 0, \\ \sum_{j=1}^{3} \lambda_j \lambda_{j+1} = 1, \end{cases} \tag{B.12}$$

in which λ_3 is a parameter, one has, with $\Delta = -3\lambda_3^2 - 4$,

$$\lambda_1 = \frac{-\lambda_3 + \sqrt{\Delta}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{-\lambda_3 - \sqrt{\Delta}}{2}.$$

This implies

$$\alpha = \alpha(\lambda_3) = \lambda_3 - \lambda_1 = \frac{3\lambda_3 - \sqrt{\Delta}}{2}, \quad \beta = \beta(\lambda_3) = \lambda_3 - \lambda_2 = \frac{3\lambda_3 + \sqrt{\Delta}}{2}. \tag{B.13}$$

Thus, if z is a common root of $|\det Q|$ and $|P|$ and $\lambda_i(z) \neq \lambda_j(z)$ for $i \neq j$ $(1 \leq i, j \leq 3)$, then, by (B.10) and (B.11),

$$(e^{\alpha L} - 1)(e^{-\beta L} - 1) = (e^{-\alpha L} - 1)(e^{\beta L} - 1),$$

which is equivalent to

$$(e^{\alpha L} - e^{\beta L})(e^{\alpha L} - 1)(e^{\beta L} - 1) = 0.$$

This implies that either $e^{\alpha L} = e^{\beta L}$, or $e^{\alpha L} = 1$, or $e^{\beta L} = 1$. Since $\lambda_1, \lambda_2, \lambda_3$ are distinct, it follows from (B.10) and (B.11) that

$$e^{\alpha L} = e^{\beta L} = 1. \tag{B.14}$$

We deduce from (B.13) that

$$3\lambda_3 \in 2\pi i \mathbb{Z}/L.$$

Since $\lambda_3^3 + \lambda_3 = -iz$, it follows that z is real. The proof is complete. ∎

Lemma B.3. *There exist $c, C > 0$ and $m_0 \in \mathbb{N}_*$ such that*

(1) *for $m \in \mathbb{Z}$ with $|m| \geq m_0$, we have*

$$|\det Q(z)| \geq Ce^{-c|z|^{1/3}} \quad \text{if } \Im(z) = ((2m+1)\pi/(\sqrt{3}\,L))^3;$$

(2) *for $z \in \mathbb{C}$ with $|z| \geq m_0$ and $|\Re(z)| \geq c|z|^{1/3}$, we have*

$$|\det Q(z)| \geq Ce^{-c|z|^{1/3}}.$$

Proof. For $z \in \mathbb{C}$ with large $|z|$, denote by $\lambda_1, \lambda_2, \lambda_3$ the roots of the equation

$$\lambda^3 + \lambda = -iz,$$

with the convention $\Re(\lambda_3) \geq \max\{\Re(\lambda_1), \Re(\lambda_2)\}$, and, with $\Delta = -3\lambda_3^2 - 4$,

$$\lambda_1 = \frac{-\lambda_3 + \sqrt{\Delta}}{2} \quad \text{and} \quad \lambda_2 = \frac{-\lambda_3 - \sqrt{\Delta}}{2}.$$

This is possible since

$$\begin{cases} \lambda_1 + \lambda_2 = -\lambda_3, \\ \lambda_1\lambda_2 = 1 + \lambda_3^2. \end{cases}$$

We have

$$|\lambda_3^{-1} \det Q(z) e^{\lambda_3 L}| = |f(\lambda_3)|,$$

where

$$f(\lambda_3) := \frac{3\lambda_3 - \sqrt{\Delta}}{2\lambda_3}(e^{\frac{3\lambda_3+\sqrt{\Delta}}{2}L} - 1) - \frac{3\lambda_3 + \sqrt{\Delta}}{2\lambda_3}(e^{\frac{3\lambda_3-\sqrt{\Delta}}{2}L} - 1). \tag{B.15}$$

Since λ_3 is large, we have

$$\left(\frac{3 - i\sqrt{3}}{2}\right)^{-1} f(\lambda_3) = [1 + O(\lambda_3^{-2})](e^{\frac{3+i\sqrt{3}}{2}\lambda_3 L + O(\lambda_3^{-1})} - 1)$$
$$- [1 + O(\lambda_3^{-2})]e^{i\varphi_0}(e^{\frac{3-i\sqrt{3}}{2}\lambda_3 L + O(\lambda_3^{-1})} - 1), \tag{B.16}$$

where $\varphi_0 = \pi/3$ since $\frac{3+i\sqrt{3}}{2}/\frac{3-i\sqrt{3}}{2} = e^{i\varphi_0}$.

(1) It suffices to prove, for $z \in \mathbb{C}$ with $\Im(z) = ((2m+1)\pi/(\sqrt{3}\,L))^3$ with large $|m|$ ($m \in \mathbb{Z}$), that

$$|\lambda_3^{-1} \det Q(z) e^{\lambda_3 L}| \geq 1. \tag{B.17}$$

Assume that (B.17) does not hold. Then for some $m \in \mathbb{Z}$ with large modulus and for some $z \in \mathbb{C}$ with $\Im(z) = ((2m+1)\pi/(\sqrt{3}\,L))^3$, we have

$$|f(\lambda_3)| \leq 1.$$

Since $\Re(\lambda_3)$ is positive and large, it follows that

$$|e^{\frac{3+i\sqrt{3}}{2}\lambda_3 L}| = (1 + O(\lambda_3^{-1}))|e^{\frac{3-i\sqrt{3}}{2}\lambda_3 L}|.$$

One finds that if $\lambda_3 = a + ib$ with $a, b \in \mathbb{R}$, then

$$a \text{ is large} \quad \text{and} \quad |b| = O(\lambda_3^{-1}). \tag{B.18}$$

It follows that

$$e^{\frac{3+i\sqrt{3}}{2}\lambda_3 L} = e^{\frac{3aL}{2}} e^{i\frac{\sqrt{3}aL}{2}} e^{O(\lambda_3^{-1})} \quad \text{and} \quad e^{\frac{3-i\sqrt{3}}{2}\lambda_3 L} = e^{\frac{3aL}{2}} e^{-i\frac{\sqrt{3}aL}{2}} e^{O(\lambda_3^{-1})}.$$

Using (B.16), and the fact $|f(\lambda_3)| \le 1$ and $\Im(z) = ((2m+1)\pi/(\sqrt{3}\,L))^3$, we obtain a contradiction. Hence (B.17) holds. The proof of (1) is complete.

(2) It suffices to prove (B.17) for $z \in \mathbb{C}$ with $|z| \ge m_0$ and $|\Re(z)| \ge c|z|^{1/3}$ for some $c > 0$. This indeed follows from the fact if $|z|$ is large and $|f(\lambda_3)| \le 1$, then (B.18) holds. The proof is complete. ∎

Acknowledgements. The authors were partially supported by ANR Finite4SoS ANR-15-CE23-0007. A. Koenig is supported by a public grant overseen by the French National Research Agency (ANR) as part of the "Investissements d'Avenir" program of the Idex PSL reference ANR-10-IDEX-0001-02 PSL. H.-M. Nguyen thanks Fondation des Sciences Mathématiques de Paris (FSMP) for the Chaire d'excellence which allowed him to visit Laboratoire Jacques-Louis Lions and Mines ParisTech. This work was done during that visit.

References

[1] Ahlfors, L. V.: Complex Analysis. 3rd ed., McGraw-Hill, New York (1978) Zbl 1477.30001 MR 510197

[2] Ammar-Khodja, F., Benabdallah, A., González-Burgos, M., de Teresa, L.: Recent results on the controllability of linear coupled parabolic problems: a survey. Math. Control Related Fields **1**, 267–306 (2011) Zbl 1235.93041 MR 2846087

[3] Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. **30**, 1024–1065 (1992) Zbl 0786.93009 MR 1178650

[4] Beauchard, K., Dardé, J., Ervedoza, S.: Minimal time issues for the observability of Grushin-type equations. Ann. Inst. Fourier (Grenoble) **70**, 247–312 (2020) Zbl 1448.35316 MR 4105940

[5] Beauchard, K., Helffer, B., Henry, R., Robbiano, L.: Degenerate parabolic operators of Kolmogorov type with a geometric control condition. ESAIM Control Optim. Calc. Var. **21**, 487–512 (2015) Zbl 1311.93042 MR 3348409

[6] Beauchard, K., Koenig, A., Le Balc'h, K.: Null-controllability of linear parabolic transport systems. J. École Polytech. Math. **7**, 743–802 (2020) Zbl 1443.93030 MR 4086584

[7] Beauchard, K., Marbach, F.: Quadratic obstructions to small-time local controllability for scalar-input systems. J. Differential Equations **264**, 3704–3774 (2018) Zbl 1377.93042 MR 3741402

[8] Beauchard, K., Marbach, F.: Unexpected quadratic behaviors for the small-time local null controllability of scalar-input parabolic equations. J. Math. Pures Appl. (9) **136**, 22–91 (2020) Zbl 1436.93018 MR 4076969

[9] Beauchard, K., Miller, L., Morancey, M.: 2D Grushin-type equations: minimal time and null controllable data. J. Differential Equations **259**, 5813–5845 (2015) Zbl 1321.35098 MR 3397310

[10] Beauchard, K., Morancey, M.: Local controllability of 1D Schrödinger equations with bilinear control and minimal time. Math. Control Related Fields **4**, 125–160 (2014) Zbl 1281.93016 MR 3167929

[11] Benabdallah, A., Boyer, F., Morancey, M.: A block moment method to handle spectral condensation phenomenon in parabolic control problems. Ann. H. Lebesgue **3**, 717–793 (2020) Zbl 1453.93016 MR 4149825

[12] Bona, J. L., Sun, S. M., Zhang, B.-Y.: A nonhomogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finite domain. Comm. Partial Differential Equations **28**, 1391–1436 (2003) Zbl 1057.35049 MR 1998942

[13] Bona, J. L., Sun, S. M., Zhang, B.-Y.: A non-homogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finite domain. II. J. Differential Equations **247**, 2558–2596 (2009) Zbl 1181.35228 MR 2568064

[14] Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. Geom. Funct. Anal. **3**, 107–156 (1993) Zbl 0787.35097 MR 1209299

[15] Boussinesq, J.: Essai sur la théorie des eaux courantes. Mémoires présentés par divers savants, Acad. Sci. Inst. Nat. France **23**, 666–680 (1877)

[16] Burq, N., Gérard, P.: Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. C. R. Acad. Sci. Paris Sér. I Math. **325**, 749–752 (1997) Zbl 0906.93008 MR 1483711

[17] Capistrano-Filho, R. A., Pazoto, A. F., Rosier, L.: Internal controllability of the Korteweg–de Vries equation on a bounded domain. ESAIM Control Optim. Calc. Var. **21**, 1076–1107 (2015) Zbl 1331.35302 MR 3395756

[18] Cerpa, E.: Exact controllability of a nonlinear Korteweg–de Vries equation on a critical spatial domain. SIAM J. Control Optim. **46**, 877–899 (2007) Zbl 1147.93005 MR 2338431

[19] Cerpa, E.: Control of a Korteweg–de Vries equation: a tutorial. Math. Control Related Fields **4**, 45–99 (2014) Zbl 1281.93018 MR 3191303

[20] Cerpa, E., Crépeau, E.: Boundary controllability for the nonlinear Korteweg–de Vries equation on any critical domain. Ann. Inst. H. Poincaré C Anal. Non Linéaire **26**, 457–475 (2009) Zbl 1158.93006 MR 2504039

[21] Coron, J.-M.: On the small-time local controllability of a quantum particle in a moving one-dimensional infinite square potential well. C. R. Math. Acad. Sci. Paris **342**, 103–108 (2006) Zbl 1082.93002 MR 2193655

[22] Coron, J.-M.: Control and Nonlinearity. Math. Surveys Monogr, 136, Amer. Math. Soc., Providence, RI (2007) Zbl 1140.93002 MR 2302744

[23] Coron, J.-M.: Some open problems on the control of nonlinear partial differential equations. In: Perspectives in Nonlinear Partial Differential Equations, Contemp. Math. 446, Amer. Math. Soc., Providence, RI, 215–243 (2007) Zbl 1200.93018 MR 2376661

[24] Coron, J.-M., Crépeau, E.: Exact boundary controllability of a nonlinear KdV equation with critical lengths. J. Eur. Math. Soc. **6**, 367–398 (2004) Zbl 1061.93054 MR 2060480

[25] Coron, J.-M., Nguyen, H.-M.: Optimal time for the controllability of linear hyperbolic systems in one-dimensional space. SIAM J. Control Optim. **57**, 1127–1156 (2019) Zbl 1418.35259 MR 3932617

[26] Duprez, M., Koenig, A.: Control of the Grushin equation: non-rectangular control region and minimal time. ESAIM Control Optim. Calc. Var. **26**, art. 3, 18 pp. (2020) Zbl 1447.93025 MR 4050579

[27] Ervedoza, S., Zuazua, E.: A systematic method for building smooth controls for smooth data. Discrete Contin. Dynam. Systems Ser. B **14**, 1375–1401 (2010) Zbl 1219.93011 MR 2679646

[28] Glass, O., Guerrero, S.: Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit. Asymptot. Anal. **60**, 61–100 (2008) Zbl 1160.35063 MR 2463799

[29] Kato, T.: On the Cauchy problem for the (generalized) Korteweg–de Vries equation. In: Studies in Applied Mathematics, Adv. Math. Suppl. Stud. 8, Academic Press, New York, 93–128 (1983) Zbl 0549.34001 MR 759907

[30] Korteweg, D. J., de Vries, G.: On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos. Mag. (5) **39**, 422–443 (1895) Zbl 26.0881.02 MR 3363408

[31] Linares, F., Ponce, G.: Introduction to Nonlinear Dispersive Equations. 2nd ed., Universitext, Springer, New York (2015) Zbl 1310.35002 MR 3308874

[32] Marbach, F.: An obstruction to small-time local null controllability for a viscous Burgers' equation. Ann. Sci. École Norm. Sup. (4) **51**, 1129–1177 (2018) Zbl 1415.93051 MR 3942039

[33] Miura, R. M.: The Korteweg–de Vries equation: a survey of results. SIAM Rev. **18**, 412–459 (1976) Zbl 0333.35021 MR 404890

[34] Molinet, L., Ribaud, F.: On the low regularity of the Korteweg–de Vries–Burgers equation. Int. Math. Res. Notices **2002**, 1979–2005 Zbl 1031.35126 MR 1918236

[35] Nguyen, H.-M., Squassina, M.: Fractional Caffarelli–Kohn–Nirenberg inequalities. J. Funct. Anal. **274**, 2661–2672 (2018) Zbl 1393.46027 MR 3771839

[36] Pazoto, A. F.: Unique continuation and decay for the Korteweg–de Vries equation with localized damping. ESAIM Control Optim. Calc. Var. **11**, 473–486 (2005) Zbl 1148.35348 MR 2148854

[37] Perla Menzala, G., Vasconcellos, C. F., Zuazua, E.: Stabilization of the Korteweg–de Vries equation with localized damping. Quart. Appl. Math. **60**, 111–129 (2002) Zbl 1039.35107 MR 1878262

[38] Rosier, L.: Exact boundary controllability for the Korteweg–de Vries equation on a bounded domain. ESAIM Control Optim. Calc. Var. **2**, 33–55 (1997) Zbl 0873.93008 MR 1440078

[39] Rosier, L.: Control of the surface of a fluid by a wavemaker. ESAIM Control Optim. Calc. Var. **10**, 346–380 (2004) Zbl 1094.93014 MR 2084328

[40] Rosier, L., Zhang, B.-Y.: Control and stabilization of the Korteweg–de Vries equation: recent progresses. J. Syst. Sci. Complex. **22**, 647–682 (2009) Zbl 1300.93091 MR 2565262

[41] Rudin, W.: Real and Complex Analysis. 2nd ed., McGraw-Hill, New York (1987) Zbl 0278.26001 MR 0344043

[42] Russell, D. L., Zhang, B. Y.: Exact controllability and stabilizability of the Korteweg–de Vries equation. Trans. Amer. Math. Soc. **348**, 3643–3672 (1996) Zbl 0862.93035 MR 1360229

[43] Sussmann, H. J.: A general theorem on local controllability. SIAM J. Control Optim. **25**, 158–194 (1987) Zbl 0629.93012 MR 872457

[44] Tao, T.: Nonlinear Dispersive Equations. CBMS Reg. Conf. Ser. Math. 106, Amer. Math. Soc., Providence, RI (2006) Zbl 1106.82376 MR 2233925

[45] Tenenbaum, G., Tucsnak, M.: New blow-up rates for fast controls of Schrödinger and heat equations. J. Differential Equations **243**, 70–100 (2007) Zbl 1127.93016 MR 2363470

[46] Whitham, G. B.: Linear and Nonlinear Waves. Wiley-Interscience, New York (1974) Zbl 0373.76001 MR 0483954