Lack of local controllability for a water-tank system when the time is not large enough

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November 26, 2022

Abstract

We consider the small-time local controllability property of a water tank modeled by 1D Saint-Venant equations, where the control is the acceleration of the tank. It is known from the work of Dubois et al. that the linearized system is not controllable. Moreover, concerning the linearized system, they showed that a traveling time T^* is necessary to bring the tank from one position to another for which the water is still at the beginning and at the end. Concerning the nonlinear system, Coron showed that local controllability around equilibrium states holds for a time large enough. In this paper, we show that for the local controllability of the nonlinear system around the equilibrium states, the necessary time is at least $2T^*$ even for the tank being still at the beginning and at the end. The key point of the proof is a coercivity property for the quadratic approximation of the water-tank system.

MSC 2020 classification: 93B05, 93C10, 93C20, 35L40, 35L60

1 Introduction

1.1 Statement of the main result

We consider a water tank with a length $L > 0$ in the time interval $(0, T)$ modeled by the following 1D Saint-Venant system (see fig. 1):

$$\begin{cases} \partial_t H + \partial_x (vH) = 0 & \text{for } (t,x) \in (0,T) \times (0,L), \\ \partial_t v + \partial_x \left(gH + \frac{v^2}{2} \right) = -u(t) & \text{for } (t,x) \in (0,T) \times (0,L), \\ v(t,0) = v(t,L) = 0 & \text{for } t \in (0,T), \end{cases} \tag{1}$$

and

$$\ddot{D}(t) = u(t) \text{ for } t \in (0,T). \tag{2}$$

Here H denotes the height of the water, v is the horizontal velocity field of the water, u is the acceleration that is imposed on the tank, D is the position of the tank, g is the gravity. Given $H_{\text{eq}} > 0$, one can easily check that $(H_{\text{eq}}, 0)$ is a solution of (1) and thus is an equilibrium of (1).

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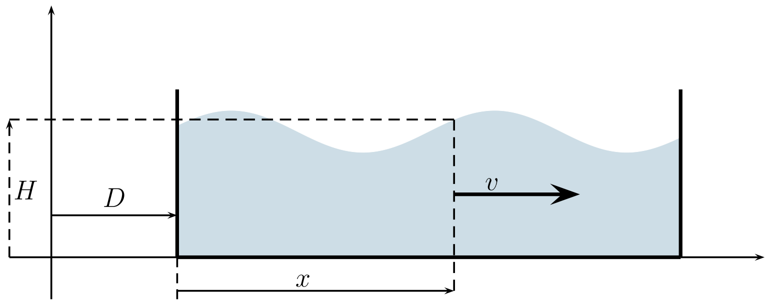

Figure 1: Water tank problem

The well-posedness of the system (1) will be discussed in proposition 23. In this article, we are interested in the *local controllability* of this system (see, e.g., [12, Section 1] for a definition). We prove that the system (1)–(2) is not locally-controllable around $(H, v) = (H_{\text{eq}}, 0)$ and $(D, \dot{D}) = (0, 0)$:

Theorem 1. *Let $L > 0$, $g > 0$, and $H_{\text{eq}} > 0$. Set*

$$T_* := \frac{L}{\sqrt{H_{\text{eq}} g}}. \tag{3}$$

Let $T \in (T_*, 2T_*)$. There exists $\eta > 0$ such that for every $u \in C^0([0, T])$ with $u(0) = 0$ and $\|u\|_{C^0([0,T])} < \eta$, if the solution $(H, v) \in \left[C^1([0, T] \times [0, L])\right]^2$ of the water-tank system (1) with the initial data

$$H(0, \cdot) = H_{\text{eq}} \quad \textit{and} \quad v(0, \cdot) = 0, \tag{4}$$

satisfies

$$H(T, \cdot) = H_{\text{eq}} \quad \textit{and} \quad v(T, \cdot) = 0, \tag{5}$$

and the solution D of (2) satisfies

$$\dot{D}(T) = \dot{D}(0) = 0, \tag{6}$$

then

$$u = 0 \textit{ in } (0, T). \tag{7}$$

Conditions (4) and (5) reads "u steers the water-tank system from $(H_{\text{eq}}, 0)$ to $(H_{\text{eq}}, 0)$ at time T", while condition (6) reads "the water-tank ends with the same speed as the one it started with". As a consequence of Theorem 1, the water-tank system is not locally controllable around $(H, v) = (H_{\text{eq}}, 0)$ and $(D, \dot{D}) = (0, 0)$ for time smaller than $2T_*$ (with controls small in $C^0([0, T])$).

Remark 2.
1. The regularity required for the control u, namely C^0, might be somehow unexpected. Standard well-posedness theorems would assume the source term u (small) in C^1. The specific form of the source term ($u(t)$ instead of $u(t, x)$) is used for this point.

2. The time T_* is the time needed for waves of the linearized equation to travel from one end of the tank to the other end, as observed in [19].

3. The water-tank system (1) is a hyperbolic system. As such, there is a finite speed of propagation, and it is no surprise local-controllability fails in small time (see remark 24). The interest of this theorem is that the local controllability fails even for times larger than what the finite speed of propagation would suggest.

4. Does a similar theorem holds for the system (1) (without $(D, \dot{D})$ as part of the state)? This is an open problem, but an essential part of our method, the so-called quadratic drift, irremediably breaks down. We discuss this in remark 22.

The controllability of the water-tank system was initially considered by Dubois, Petit and Rouchon [19] where the linearized system was considered. In particular, they proved for the linearized system that, given $T > T_*$, there exists a control that steers an equilibrium $(H_{eq}, 0)$ back to itself while moving the water-tank.

Concerning the nonlinear system, the local-controllability was investigated by Coron [12] using the *return method*. More precisely, Coron proved that local controllability around equilibrium states $(H_{eq}, 0)$ for (H, v) starting with $(\dot{D}(0), D(0))$ near (s_0, D_0) and ending with $(\dot{D}(T), D(T))$ near $(s_0, D_0 + Ts_0)$ for a time T large enough. In particular, the local controllability around $(H_{eq}, 0)$ (for (H, v)) and $(0, 0)$ (for $(\dot{D}, D)$) holds for a large enough time.

Theorem 1 reveals new properties for the local controllability of the nonlinear water tank problem. First, Theorem 1 reveals that for $T_* < T < 2T_*$, contrary to the linearized system, one cannot steer an equilibrium $H(0, x) = H_{eq}$, $v(0, x) = 0$ back to itself if the water-tank ends with the same speed as the one it started with (except for the trivial trajectory where $u = 0$). Theorem 1 also points out that the local controllability around $(H_{eq}, 0)$ (for (H, v)) and $(0, 0)$ (for $(\dot{D}, D)$) holds but with at time larger than or equal to $2T^*$.

The optimal time for the boundary controllability of hyperbolic systems have been studied extensively, see [18, 16, 17], where the controls are on one side. This is different from the water tank problem which can be seen as a boundary control problem where the controls are given on two sides, see eqs. (34) and (35). Moreover, the controls for the water tank problem the controls are required to be the same on both side, see eq. (35). This rigidity condition yields new phenomena and obstructions that require new ingredients to describe.

1.2 The main ideas of the proof and the organization of the paper

Using standard scaling arguments (see for instance [12, Section 2]), namely setting

$$H^*(t, x) := \frac{1}{H_{eq}} H\left(\frac{L}{\sqrt{H_{eq} g}} t, Lx\right);$$
$$v^*(t, x) := \frac{1}{\sqrt{H_{eq} g}} v\left(\frac{L}{\sqrt{H_{eq} g}} t, Lx\right),$$

we may assume that $L = 1$, $g = 1$, and $H_{eq} = 1$ and this will be assumed from now on. Note that in this case, T_* defined in theorem 1 is $T_* = 1$.

The proof has its root in the *power series expansion method*, see, e.g., [14] and [11, Chapter 8]: since the linearized system does not give enough information to conclude about the local-controllability of (1), we consider the second-order approximation. Indeed, the linearized system of (1) around the equilibrium $(1, 0)$ is

$$\begin{cases} \partial_t h_1 + \partial_x v_1 = 0 & \text{for } (t, x) \in (0, T) \times (0, 1), \\ \partial_t v_1 + \partial_x h_1 = -u(t) & \text{for } (t, x) \in (0, T) \times (0, 1), \\ v_1(t, 0) = v_1(t, L) = 0 & \text{for } t \in (0, T). \end{cases} \tag{8}$$

Simple computations prove that if $h_1(0, x) = 0$ and $v_1(0, x) = 0$, then $h_1(t, 1 - x) = -h_1(t, x)$ and $v_1(t, 1 - x) = v_1(t, x)$ whatever u is. Thus, the linearized system is not controllable. As usual, the second order approximation system is given as follows

$$\begin{cases} \partial_t h_2 + \partial_x v_2 = -\partial_x(h_1 v_1) & \text{for } (t,x) \in (0,T)\times(0,1), \\ \partial_t v_2 + \partial_x h_2 = -\partial_x\left(\frac{v_1^2}{2}\right) & \text{for } (t,x) \in (0,T)\times(0,1), \\ v_2(t,0) = v_2(t,L) = 0 & \text{for } t \in (0,T). \end{cases} \tag{9}$$

The main idea is to prove that if a control steers the linearized system from 0 to 0, this second order always lies in some half-space, at least when $T < 2T^*$. More precisely, for $T^* < T < 2T^*$, we prove that for well-chosen functions ϕ, ψ, there exists $c > 0$ such that for every control u that steers the linearized system from 0 to 0 and such that $\int_0^T u(s)\,\mathrm{d}s = 0$, with

$$U(t) := \int_0^t u(s)\,\mathrm{d}s, \tag{10}$$

we have

$$(h_2(T,\cdot),\phi) + (v_2(T,\cdot),\psi) \geq c\|U\|_{L^2}^2, \tag{11}$$

This means that the quadratic approximation of the water-tank system cannot be steered into the half-space $\{(h,v) \in L^2(0,1)^2, (h,\phi) + (v,\psi) < 0\}$. The rest of the proof consists in estimating the difference between the quadratic approximation and the nonlinear system in an appropriate way so that one can reach the optimal control space.

The paper is organized as follows:

1. in section 2, we characterize the controls that steer the linearized system from 0 to 0;
2. in section 3, we analyse the second-order term, and prove that it satisfies a "conditional H^{-1}-coercivity" property;
3. in section 4, we study the nonlinear system, and in particular we prove that the error between the nonlinear solution and the second-order approximation cannot counter the positivity of the second-order term.

1.3 Bibliographical comments

Our proof relies on the positivity of a scalar product of the quadratic approximation of the water-tank system (1). This kind of phenomenon was the heart of several lack of small time local controllability results for systems modeled by partial differential equations. Concerning examples in finite dimensional system, we refer to Beauchard and Marbach's paper [5], and the references therein.

The quadratic obstructions for small-time local controllability was previously observed for the Schrödinger equation with bilinear control [13, 7, 10], the viscous Burgers equation [22], nonlinear heat equations [6] and a KdV system [15] where the speed of the propagation is infinite. All these results share the same core idea: the scalar product of the second-order approximation with appropriate test functions enjoys a coercivity property. Let us detail a little each of these cases.

For the Schrödinger equation with bilinear control, the existing results relies heavily on explicit computation using the eigenfunctions and eigenvalues of the operator $-\partial_x^2$. Note that in Coron's result [13] as well as Beauchard and Morancey's result [7], the equivalent of our coercivity estimate (11) also has $\|U\|_{L^2}^2$ in the right-hand side, leading to a lack of small-time local controllability with controls small in L^∞-norm. Bournissou [10] also has a similar coercivity estimate, with the nth iterated integral of the control instead of U, where n depends on the structure of the potential. This leads to a lack of small-time local controllability with controls small either in $W^{-1,\infty}$ (when $n = 1$) or H^{2n-3} (when $n \geq 2$).

Marbach [22] considered a viscous Burgers equation with control $u(t)$ as a source term. The main difficulty is the fact that the kernel of the quadratic approximation does not seem to be explicitly computable in a usable form. To tackle the problem, he rescaled the equation in time to transform the "small-time" aspect of the problem into a small-viscosity problem. This allows him to compute an asymptotic expansion of the kernel of the quadratic approximation of the viscous Burgers equation in low viscosity limit. Using this, Marbach succeeds in disproving the small-time local controllability with controls small in L^2-norm. A striking feature of this result is the equivalent of our coercivity estimate (11) has the $H^{-5/4}$-norm of the control in the right-hand side, a *noninteger* Sobolev norm.

Beauchard and Marbach [6] considered a class of nonlinear heat equation. They exhibit a range of phenomena. For instance, for some nonlinearities, they prove a coercivity estimate with the H^{-s}-norm of the control for some $s > 0$ that depends on the nonlinearity and that can be fractional. Also, for other nonlinearities, the quadratic term can actually *help* recover the small-time local controllability. This is the first example in which the quadratic term gives the local controllability result.

Concerning the KdV equations [15], we proved that the KdV equation with Dirichlet boundary conditions and Neumann boundary control on the right is not small-time locally controllable with controls small in H^1 for some critical lengths, introduced previously by Rosier [25]. This fact is surprising when compared with known results on internal controls for the corresponding KdV system for which the small time result holds (see e.g., [23]). One of the main difficulties was to characterize the controls that steers the linearized equation from 0 to 0. The analysis is based on a complete characterization of controls which bring 0 to 0 for the linearized system that involves the Paley-Wiener Theorem. The equivalent of the coercivity estimate (11) has the $H^{-2/3}$-norm of the control in the right-hand side.

The result of this paper compares to the previous ones in the following aspects:

- The control is internal, as was the case for the bilinear Schrödinger equation and the viscous Burgers equation, and unlike the KdV equation (where the control was at the boundary).
- Even if the computations are lengthy, we are able to compute the kernel of the second-order approximation in a very simple closed-form expression, which was more or less the case of the bilinear Schrödinger equation, but was not the case for the viscous Burgers equation and the KdV equation, where only an asymptotic expansion of the kernel was computed in closed form.
- We are able to disprove the small-time local controllability with controls small in C^0, which is the natural space for the known well-posedness results. This is different from some bilinear Schrödinger equations, some nonlinear heat equations, and the KdV equation, where the existing results require the control to be quite regular. It is worth noting that less regular controls can change the situation. This is done for the Schrödinger equation by Bournissou [9] where the cubic terms surprisingly help recover the local controllability even in the case where the quadratic term gives the obstruction if regular controls are used.

Finally, we note that even with infinite speed of propagation in the linear setting, there might not be small-time controllability when there is a concentration of eigenfunction [4, 3, 20] or when there is condensation of eigenvalues or eigenfunctions [1, 8] (see also references therein).

2 Preliminary properties of the linearized system

As explained in section 1.2, without loss of generality, we may assume that $g = 1$ and $L = H_{\text{eq}} = 1$. Then the linearization of the system (1) around the equilibrium $(H_{\text{eq}}, 0) = (1, 0)$ is given by the system (8).

This system can be rewritten as $\partial_t F + \mathcal{A}F = U(t)$ with $F = (h_1, v_1) \in (L^2)^2$, $U(t) = (0, -u(t))$ and $\mathcal{A}$ is the unbounded operator on $H = (L^2)^2$ with domain $D(\mathcal{A}) := H^1 \times H^1_0$ and defined by $\mathcal{A}(h, v) = (\partial_x v, \partial_x h)$. One can prove this system is well-posed thanks, e.g., to Lummer-Philips' theorem [24, Theorem 4.3].

2.1 Periodic change of variables

From now on, we denote

$$\mathbb{T} := \mathbb{R}/2\mathbb{Z}. \tag{12}$$

It is convenient to introduce the following periodic change of variables.

Definition 3. Given $F = (h, v) \in [L^2(0,1)]^2$, define $\mathcal{C}F \in L^2(\mathbb{T})$ by

$$\mathcal{C}F(x) = \begin{cases} h(x) + v(x) & \text{for } 0 < x < 1, \\ h(-x) - v(-x) & \text{for } -1 < x < 0. \end{cases} \tag{13}$$

This change of variables transforms the linearized water-tank system into a transport equation with periodic boundary conditions:

Proposition 4. *Let $(H, v) \in C^1([0,T] \times [0,1])^2$ such that $v(t,0) = v(t,1) = 0$ and denote*

$$\zeta(t,\cdot) = \mathcal{C}(H(t,\cdot), v(t,\cdot)).$$

Then

- *ζ is continuous in $[0,T] \times \mathbb{T}$ and is C^1 in $[0,T] \times (\mathbb{T} \setminus \{0,1\})$;*
- *If in addition $U \in L^\infty([0,T] \times [0,1])^2$ and*

$$\partial_t (H, v)(t,x) + \mathcal{A}(H,v)(t,x) = U(t,x) \text{ for } (t,x) \in [0,T] \times [0,1], \tag{14}$$

then

$$\partial_t \zeta(t,x) + \partial_x \zeta(t,x) = \mathcal{C}U(t,x) \text{ for every } t \geq 0 \text{ and } x \in \mathbb{T} \setminus \{0,1\}. \tag{15}$$

Proof. The fact that $\zeta = \mathcal{C}(H,v)$ is C^1 in $[0,T] \times (\mathbb{T} \setminus \{0,1\})$ is a direct consequence of the definition of $\mathcal{C}$. The continuity at $x = 0$ and $x = 1$ results from the boundary conditions $v(t,0) = v(t,1) = 0$.

The second point results from elementary computations. □

Remark 5. We can check that $\mathcal{C}$ is an isometry (up to a factor 2) from $L^2(0,1)^2$ to $L^2(\mathbb{T})$, and that if $F = (H, v) \in C^1([0,1])^2$ with $v(0) = v(1) = 0$, then $\|\mathcal{C}F\|_{W^{1,\infty}} \leq 2\|F\|_{C^1}$.

Using the characteristic method, one can obtain the following formula for the solution of (15).

Lemma 6. *Let $w \in L^2((0,T) \times \mathbb{T})$. The solution ζ of $\partial_t \zeta(t,x) + \partial_x \zeta(t,x) = w(t,x)$, $\zeta(0,x) = 0$ is*

$$\zeta(t,x) = \int_0^t w(s, x + s - t)\, \mathrm{d}s.$$

The linearized system (8) with zero initial conditions can be rewritten in $\zeta_1(u,t,x) = \mathcal{C}(h_1, v_1)(t,x)$ variable as

$$(\partial_t + \partial_x)\zeta_1(u,t,x) = u(t)\theta(x), \quad \zeta_1(u,0,\cdot) = 0, \tag{16}$$

where θ is a "square wave" function that is 2-periodic defined by

$$\theta(x)=\begin{cases}1 & \text{on } (-1,0),\\ -1 & \text{on } (0,1).\end{cases} \tag{17}$$

By Lemma 6, we have

$$\zeta_1(u,t,x)=\int_0^t u(s)\theta(x+s-t)\,\mathrm{d}s. \tag{18}$$

Remark 7. We remark that $\theta(x+1)=-\theta(x)$, thus, $\zeta_1(u,t,x+1)=-\zeta_1(u,t,x)$.

Another useful formula for ζ_1 is:

Lemma 8. *Let $u\in L^2(0,T)$, extended by 0 for $t<0$, and set $U(t):=\int_0^t u(s)\,\mathrm{d}s$. Then, for $0<x<1$ and $t>0$,*[1]

$$\zeta_1(u,t,x)=-U(t)+2\sum_{k=0}^{+\infty}(-1)^k U(t-x-k).$$

Proof. If we define $\tilde{\zeta}_1$ as the right hand side of this formula, we see that $\tilde{\zeta}_1(u,t,1)=-\tilde{\zeta}_1(u,t,0)$, so that the 1-antiperiodic extension of $\tilde{\zeta}_1$ is continuous in $(t,x)\in[0,T]\times\mathbb{T}$. Moreover, we see that for $0<x<1$ and $t>0$

$$(\partial_t+\partial_x)\tilde{\zeta}_1(u,t,x)=-u(t).$$

Thus, if we still denote by $\tilde{\zeta}_1$ the 1-antiperiodic extension of $\tilde{\zeta}_1$, we have $(\partial_t+\partial_x)\tilde{\zeta}_1=u(t)\theta(x)$. Thus, $\tilde{\zeta}_1=\zeta_1$. □

We will sometimes denote this $\zeta_1(u,t,x)$ by $\zeta_1(t,x)$, leaving the fact that it depends on u implicit. We will use similar notations for every quantities that depends on the control.

Let us finally give some estimates for ζ_1. In what follows, for $T>0$, we use the notations $L^2_tL^2_x$ and $L^\infty_tL^2_x$ as a shorthand for $L^2(0,T;L^2(\mathbb{T}))$ and $L^\infty(0,T;L^2(\mathbb{T}))$.

Proposition 9. *Let $T>0$. The solution ζ of $(\partial_t+\partial_x)\zeta=w$ satisfies:*

$$\|\zeta\|_{L^2_tL^2_x}\le C\|w\|_{L^2_tL^2_x}. \tag{19}$$

Moreover, in case the right-hand side is $w(t,x)=u(t)\theta(x)$, if we set $U(t):=\int_0^t u(s)\,\mathrm{d}s$, then the solution $\zeta_1(u,\cdot,\cdot)$ satisfies

$$\|\zeta_1(u)\|_{L^2_tL^2_x}\le C\|U\|_{L^2}. \tag{20}$$

Proof. The first inequality is standard, and is proved with the characteristic formula (lemma 6) and Cauchy-Schwarz inequality:

$$\begin{aligned}\|\zeta\|^2_{L^2_tL^2_x}&=\int_{[0,T]^3\times\mathbb{T}}\mathbb{1}_{s_1,s_2\le t}w(s_1,x+s_1-t)w(s_2,x+s_2-t)\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t\,\mathrm{d}x\\&\le\int_{[0,T]^3\times\mathbb{T}}w(s_1,x+s_1-t)^2\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t\,\mathrm{d}x\\&=\int_{[0,T]^3\times\mathbb{T}}w(s_1,x')^2\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t\,\mathrm{d}x',\end{aligned}$$

where we also used the change of variables $x'=x+s_1-t$. This implies the claimed estimate (19).

The second estimate is a direct consequence of lemma 8. □

[1] Note that with this extension of u, we have for $t\le 0$, $U(t)=0$, so that there is only a finite number of non-zero terms in the sum.

2.2 Control of the linearized system

We next discuss control properties for the linearized systems. We give a controllability result when the target is 1-antiperiodic and we characterize the controls that steers 0 to 0. We begin with

Lemma 10. *Let $T > 1$. For any $\zeta_T \in H^1(\mathbb{T})$ that is 1-anti-periodic (i.e., $\zeta_T(x+1) = -\zeta_T(x)$), there exists a control $u \in L^2(0,T)$ such that the solution ζ of the linear equation (16) with initial condition 0 satisfies $\zeta(T,\cdot) = \zeta_T$. Moreover, this control u can be chosen such that $\int_0^T u(t)\,\mathrm{d}t = 0$ and, if we set $U(t) := \int_0^t u(s)\,\mathrm{d}s$, such that*

$$\|U\|_{L^2(0,T)} \leq C\|\zeta_T\|_{L^2(\mathbb{T})}$$

for some C independent of ζ_T.

Proof. We construct the control using the so-called *flatness method*. The main point, inspired by Dubois, Petit, and Rouchon [19, Section 3.4], is that if $\varphi : \mathbb{R} \to \mathbb{R}$ is in $H^1(\mathbb{R})$, then the function $\zeta : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ defined by

$$\zeta(t,x) = \begin{cases} 2\varphi\left(t - x + \frac{1}{2}\right) - \varphi\left(t + \frac{1}{2}\right) - \varphi\left(t - \frac{1}{2}\right) & \text{if } 0 < x < 1 \\ -2\varphi\left(t - x - \frac{1}{2}\right) + \varphi\left(t + \frac{1}{2}\right) + \varphi\left(t - \frac{1}{2}\right) & \text{if } -1 < x < 0 \end{cases}$$

satisfies $(\partial_t + \partial_x)\zeta(t,x) = u(t)\theta(x)$ with $u(t) := -\varphi'(t+1/2) - \varphi'(t-1/2)$. We aim to construct a function φ such that the trajectory associated by this formula goes from 0 at time 0 to ζ_T at time T.

To construct φ, for $T - 1/2 < x < T + 1/2$, we set $\varphi(x) := \zeta_T(T - x + 1/2)/2$, and we extend this as a function in $H^1(\mathbb{R})$, which is still denoted by φ such that $\varphi = 0$ in $(-\infty, 1/2]$. This extension can be done so that $\zeta_T \in L^2(\mathbb{T}) \mapsto \varphi \in L^2(\mathbb{R})$ is linear and continuous.

The first condition ensures that for $0 < x < 1$, the corresponding trajectory ζ satisfies $\zeta(T,x) = \zeta_T(x)$. Since ζ_T is 1-antiperiodic, we also have $\zeta(T,x) = \zeta_T(x)$ for $-1 < x < 0$. The fact that φ is zero on $[-1/2, 1/2]$ ensures that $\zeta(0,\cdot) = 0$.

The corresponding control is $u(t) = -\varphi'(t+1/2) - \varphi'(t-1/2)$. Thus,

$$\begin{aligned}\int_0^T u(t)\,\mathrm{d}t &= -\varphi\left(\frac{1}{2}\right) - \varphi\left(-\frac{1}{2}\right) + \varphi\left(T + \frac{1}{2}\right) + \varphi\left(T - \frac{1}{2}\right) \\ &= -0 - 0 + \zeta_T(1)/2 + \zeta_T(0)/2.\end{aligned}$$

Since ζ_T is assumed to be 1-antiperiodic, we do have $\int_0^T u(t)\,\mathrm{d}t = 0$.

The last thing we have to prove is the estimate. We have $U(t) = \int_0^t u(s)\,\mathrm{d}s = -\varphi(t+1/2)/2 - \varphi(t-1/2)/2$, thus, $\|U\|_{L^2(0,T)} \leq 2\|\varphi\|_{L^2(-1/2,T+1/2)} \leq C\|\zeta_T\|_{L^2}$. □

We now study the controls that steer 0 to 0. We only prove the following condition is necessary, which is all we need, but we could also prove that it is also sufficient.

Proposition 11. *Let $T \in (1,2)$ and let $u \in L^2(0,T)$ such that the solution $\zeta_1(u,\cdot,\cdot)$ of $(\partial_t - \partial_x)\zeta_1(u,t,x) = u(t)\theta(x)$, $\zeta_1(u,0,\cdot) = 0$ satisfies $\zeta_1(u,T,\cdot) = 0$. Then*

$$u(t) = 0 \text{ for } t \in (T-1, 1) \quad \text{and} \quad u(t+1) = u(t) \text{ for } t \in (0, T-1). \tag{21}$$

Remark 12. One control that moves the water tank (with the tank ending with the same speed it started with) in time $T = 1$ is $u(t) = \delta_0'(t) + \delta_1'(t)$. In some sense, all controls that steer 0 to 0 are a regularization of this "optimal time" control.

Proof. We use the formula for ζ_1 given by lemma 8. Since $1 < T < 2$, $U(T - x - k)$ is zero whenever $k \geq 2$ and $0 < x < 1$. Hence, for $0 < x < 1$

$$\zeta_1(T, x) = -U(T) + 2U(T - x) - 2U(T - x - 1).$$

Since $\zeta_1(T, x) = 0$, by differentiating in x, we get that for $0 < x < 1$,

$$u(T - x) = u(T - x - 1).$$

If $0 < t < T - 1$, we choose $x = T - t - 1$. This proves that $u(t + 1) = u(t)$ as claimed. If $T - 1 < t < 1$, we choose $x = T - t$, which gives $u(t) = u(t - 1)$. But $u(t - 1) = 0$ (we extended u by 0 on $(-\infty, 0)$), which proves that $u(t) = 0$. □

3 Second-order approximation for the nonlinear system system

3.1 Periodic change of variables

In this section, we deal with the second order approximation system given by (9). Set

$$\zeta_2 \coloneqq \mathcal{C}(h_2, v_2) \tag{22}$$

Then

$$(\partial_t + \partial_x)\zeta_2(u, t, x) = w_1(u, t, x) \coloneqq -\,\mathcal{C}(\partial_x(h_1 v_1), \partial_x(v_1^2/2)). \tag{23}$$

Again, we will leave the fact that ζ_2, w_1, etc., depend on u implicit. We want to write w_1 as a function of ζ_1. First, using the definition of $\mathcal{C}$,

$$w_1(t, x) = \begin{cases} -\partial_x(h_1 v_1 + v_1^2/2)(t, x) & \text{for } 0 < x < 1 \\ -\partial_x(h_1 v_1 - v_1^2/2)(t, -x) & \text{for } -1 < x < 0. \end{cases}$$

We compute w_1 in term of ζ_1. We have

$$\begin{aligned} h_1(t, x) &= \frac{1}{2}(\zeta_1(t, x) + \zeta_1(t, -x)) \\ v_1(t, x) &= \frac{1}{2}(\zeta_1(t, x) - \zeta_1(t, -x)). \end{aligned}$$

So,

$$\begin{aligned} h_1 v_1(t, x) &= \frac{1}{4}(\zeta_1^2(t, x) - \zeta_1^2(t, -x)) \\ \frac{1}{2} v_1^2(t, x) &= \frac{1}{8}(\zeta_1(t, x) - \zeta_1(t, -x))^2, \end{aligned}$$

thus,

$$\begin{aligned} h_1 v_1(t, x) + \frac{1}{2} v_1^2(t, x) &= \frac{1}{8}(3\zeta_1^2(t, x) - 2\zeta_1(t, x)\zeta_1(t, -x) - \zeta_1^2(t, -x)) \\ h_1 v_1(t, x) - \frac{1}{2} v_1^2(t, x) &= \frac{1}{8}(\zeta_1^2(t, x) + 2\zeta_1(t, x)\zeta_1(t, -x) - 3\zeta_1^2(t, -x)). \end{aligned}$$

Finally, denoting $r_1(t,x) = (3\zeta_1^2(t,x) - 2\zeta_1(t,x)\zeta_1(t,-x) - \zeta_1^2(t,-x))/8$, we write this as

$$h_1 v_1(t,x) + \frac{1}{2} v_1^2(t,x) = r_1(t,x)$$
$$h_1 v_1(t,x) - \frac{1}{2} v_1^2(t,x) = -r_1(t,-x).$$

Finally, the right-hand side of (23) is

$$w_1(t,x) = \begin{cases} -\partial_x r_1(t,x) & \text{for } 0 < x < 1 \\ -\partial_x r_1(t,x) & \text{for } -1 < x < 0. \end{cases}$$

These computations are not specific to the case of the right-hand side of the second-order equation (23), but are valid whenever $w = -\mathcal{C}(\partial_x(hv), \partial_x(v^2/2))$ and $\zeta = \mathcal{C}(h,v)$.

We summarize these computations in the next lemma.

Lemma 13. *Let Q the quadratic form on $\mathbb{R}^2$ defined by $Q(a,b) := (3a^2 - 2ab - b^2)/8$. Let $\zeta = \mathcal{C}(h,v)$ for some $(h,v) \in H^1(0,1) \times H^1_0(0,1)$. Set*

$$w(x) := \mathcal{C}(-\partial_x(hv), -\partial_x(v^2/2)) \quad \text{and} \quad r(x) := Q(\zeta(x), \zeta(-x)).$$

Then

$$w(x) = -\partial_x r(x).$$

In case $\zeta(x) = \zeta_1(u,t,x)$, we will denote accordingly $w(x)$ by $w_1(u,t,x)$ and $r(x)$ by $r_1(u,t,x)$.

Remark 14. We recall that $\zeta_1(t,x+1) = -\zeta_1(t,x)$, so that $r_1(t,x+1) = r_1(t,x)$. So, w_1, as well as ζ_2 is 1-periodic in x.

3.2 Kernel for ζ_2

In this section, we express ζ_2 (or more precisely scalar products of ζ_2) via a kernel that we compute explicitly. For $a, b \in \mathbb{R}$, we denote

$$a \vee b := \max\{a,b\} \quad \text{and} \quad a \wedge b := \min\{a,b\}. \tag{24}$$

We begin with

Lemma 15. *Let ϕ be a 1-periodic C^1 function. Let q be the bilinear symmetric form on $\mathbb{R}^2$ associated to the quadratic form Q defined in lemma 13, i.e., $q(a,b,a',b') = (3aa' - ab' - a'b - bb')/8$. Define $K_t = K_t(\phi)$ by*

$$K_t(s_1,s_2) := \int_\Omega \phi'(t_1 + t - s_1 \vee s_2) q\big(\theta(t_1 - |s_2 - s_1|), \theta(t_2 - |s_2 - s_1|), \theta(t_1), \theta(t_2)\big)\, \mathrm{d}t_1\, \mathrm{d}t_2, \tag{25}$$

where $\Omega = \{2(s_1 \vee s_2 - t) < t_1 + t_2 < 0, 0 < t_1 - t_2 < 2\}$. Let $u \in L^2(0,T)$ and let $\zeta_2(u,\cdot,\cdot)$ be the second-order correction for the water-tank system, i.e., the solution of $(\partial_t + \partial_x)\zeta_2(u,t,x) = w_1(u,t,x)$, $\zeta_2(u,0,\cdot) = 0$ (where w_1 was defined in lemma 13). Then,

$$(\zeta_2(u,t,\cdot), \phi)_{L^2(\mathbb{T})} = \int_{[0,t]^2} K_t(s_1,s_2) u(s_1) u(s_2)\, \mathrm{d}s_1\, \mathrm{d}s_2.$$

Proof. As usual, we will denote $\zeta_2(u,t,x)$ by $\zeta_2(t,x)$, leaving the fact that it depends on u implicit.

This is a mostly straightforward computations using the characteristics formula. Since ζ_2 satisfies the equation $(\partial_t+\partial_x)\zeta_2(t,x) = w_1(t,x)$ with $\zeta_2(0,\cdot)=0$, then we have according to the characteristics formula:

$$\zeta_2(t,x) = \int_0^t w_1(s, s+x-t)\,\mathrm{d}s.$$

Since $w_1(s,x) = -\partial_x r_1(s,x)$, integrating by parts, we have

$$(\zeta_2(t,\cdot),\phi)_{L^2(\mathbb{T})} = -\int_{\mathbb{T}\times[0,t]} \phi(x)\partial_x r_1(s,x+s-t)\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{T}\times[0,t]} \phi'(x) r_1(s,x+s-t)\,\mathrm{d}x\,\mathrm{d}s.$$

Since the integrand is 1-periodic (r_1 is according to remark 14, and we assumed that ϕ is 1-periodic), we rewrite this as

$$(\zeta_2(t,\cdot),\phi)_{L^2(\mathbb{T})} = 2\int_{[0,1]\times[0,t]} \phi'(x) r_1(s,x+s-t)\,\mathrm{d}x\,\mathrm{d}s. \tag{26}$$

Recall that if Q is a quadratic form on $\mathbb{C}^d$ and q is its associated bilinear form, Fubini's theorem implies that for any compact subset X of $\mathbb{R}^n$ and $f : X \to \mathbb{C}^d$ measurable bounded, we have $Q(\int_X f(s)\,\mathrm{d}s) = \int_{X^2} q(f(s_1), f(s_2))\,\mathrm{d}s_1\,\mathrm{d}s_2$. Then, using the fact that $r_1(s,x) = Q(\zeta_1(s,x),\zeta_1(s,-x))$ and $\zeta_1(s,x) = \int_0^s u(s')\theta(x+s'-s)\,\mathrm{d}s'$, we get

$$\begin{aligned} r_1(s,x) &= \int_{[0,s]^2} q(u(s_1)\theta(x+s_1-s), u(s_1)\theta(-x+s_1-s), u(s_2)\theta(x+s_2-s), u(s_2)\theta(-x+s_2-s))\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &= \int_{[0,s]^2} u(s_1)u(s_2) q(\theta(x+s_1-s), \theta(-x+s_1-s), \theta(x+s_2-s), \theta(-x+s_2-s))\,\mathrm{d}s_1\,\mathrm{d}s_2. \end{aligned}$$

Plugging this into equation (26), we get that the formula $(\zeta_2(t,\cdot),\phi) = \int_{[0,t]^2} K_t(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$ holds with

$$\begin{aligned} K_t(s_1,s_2) &= 2\int_{[0,1]\times[0,t]} \mathbb{1}_{s_1,s_2\leq s}\phi'(x) q(\theta(x+s_1-t), \theta(-x+s_1-2s+t), \theta(x+s_2-t), \theta(-x+s_2-2s+t))\,\mathrm{d}x\,\mathrm{d}s \\ &= 2\int_{[0,1]\times[s_1\vee s_2,t]} \phi'(x) q(\theta(x+s_1-t), \theta(-x+s_1-2s+t), \theta(x+s_2-t), \theta(-x+s_2-2s+t))\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Since the integrand is 1-periodic in x, the change of variables $x' = x+s-t$ gives

$$K_t(s_1,s_2) = 2\int_{[0,1]\times[s_1\vee s_2,t]} \phi'(x-s+t) q(\theta(x+s_1-s), \theta(-x+s_1-s), \theta(x+s_2-s), \theta(-x+s_2-s))\,\mathrm{d}x\,\mathrm{d}s.$$

We see from this expression and the symmetry of q that $K_t(s_1,s_2) = K_t(s_2,s_1)$. So, to simplify the notation, we assume that $s_2 = s_1\vee s_2$ and $s_1 = s_1\wedge s_2$. Then, the change of variables $t_1 = x+s_2-s$, $t_2 = -x+s_2-s$, that satisfies $\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\,\mathrm{d}t_1\,\mathrm{d}t_2$ and $x-s+t = t_1-s_2+t$ proves

$$K_t(s_1,s_2) = \int_\Omega \phi'(t_1-s_2+t) q(\theta(t_1+s_1-s_2), \theta(t_2+s_1-s_2), \theta(t_1), \theta(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2,$$

where Ω is the image of $[0,1]\times[s_2,t]$. Since $x = (t_1-t_2)/2$ and $s_2 - s = (t_1+t_2)/2$, $\Omega = \{0 < t_1-t_2 < 2, 2(s_2-t) < t_1+t_2 < 0\}$. Since we swapped s_1 and s_2 so that $s_2 = s_1\vee s_2$ and $s_1 = s_1\wedge s_2$, this proves the lemma. □

Proposition 16. *Let ϕ be a C^1 1-periodic function and $t \in (0,2)$. The kernel K_t defined in lemma 15 is symmetric and for every $0 < s_1, s_2 < t$ such that $1 < s_2 - s_1$, we have $K_t(s_1, s_2) = -K_t(s_1 + 1, s_2)$. Moreover, for $0 < s_1 < s_2 < t$ and $s_2 - s_1 < 1$, we have*

$$2K_t(s_1,s_2) = \begin{cases} \displaystyle\int_{-2t+2s_2}^{0} \phi(s+t-s_2)\,\mathrm{d}s + 2(t-s_2)\phi(t-s_2) - 4(t-s_2)\phi(t-s_1) \\ \hfill \text{if } 2t-1 < s_1+s_2 < 2t \\ \displaystyle\int_{s_2-s_1}^{2-2t+s_2+s_1} \phi(s+t-s_2)\,\mathrm{d}s + (-1+4t-3s_2-s_1)\phi(t-s_2) - (1+2t-3s_2+s_1)\phi(t-s_1) \\ \hfill \text{if } 2t-2 < s_1+s_2 < 2t-1 \\ \displaystyle\int_{2-2t+2s_2}^{0} \phi(s+t-s_2)\,\mathrm{d}s + (1+2t-2s_2)\phi(t-s_2) - (-1+4t-4s_2)\phi(t-s_1) \\ \hfill \text{if } 2t-3 < s_1+s_2 < 2t-2 \\ \displaystyle\int_{s_2-s_1}^{4-2t+s_2+s_1} \phi(s+t-s_2)\,\mathrm{d}s + (-2+4t-3s_2-s_1)\phi(t-s_2) - (2+2t-3s_2+s_1)\phi(t-s_1) \\ \hfill \text{if } 2t-4 < s_1+s_2 < 2t-3 \end{cases} \tag{27}$$

Proof. First, we see from the expression of K_t given in lemma 15 (or from its proof) that $K_t(s_1,s_2) = K_t(s_2,s_1)$. Moreover, if $0 < s_1 < s_2 - 1 < s_2 < t$, we see that setting $s_1' = s_1 + 1$, $s_2' = s_2$, we have $s_1 \vee s_2 = s_1' \vee s_2'$ and that the integration set in formula (25) is the same for $K_t(s_1,s_2)$ and $K_t(s_1', s_2')$. Then, using the fact that $\theta(x+1) = -\theta(x)$ and the bilinearity of q, we get that $K_t(s_1+1, s_2) = -K_t(s_1,s_2)$. Thus, we only need to compute $K_t(s_1,s_2)$ when $0 < s_2 - s_1 < 1$.

Since $\theta(x)$ only takes the value 1 and -1 the term $q(\theta(t_1 - s_2 + s_1), \theta(t_2 - s_2 + s_1), \theta(t_1), \theta(t_2))$ only takes a finite number of values. To simplify notations, we set $\sigma = |s_2 - s_1|$, $\tau = t - s_2$ and

$$\alpha_\sigma(t_1,t_2) = q(\theta(t_1 - \sigma), \theta(t_2 - \sigma), \theta(t_1), \theta(t_2)).$$

The proof then consists in identifying which values α_σ takes and on which subsets of Ω. Then, we integrate $\int \phi'(t_1 + \tau)$ on these sets and sum everything with the right coefficient.

We remark that if a, b, a', b' are equal to ± 1, then $q(a,b,a',b')$ is equal to 0 or $\pm 1/2$. Indeed, $q(1,1,1,1) = 0$, $q(1,-1,1,1) = 1/2$, $q(1,-1,1,-1) = 1/2$ and we get the other values using the bilinearity and the symmetry of q.

Remark that α_σ can only change value when t_1 or t_2 crosses the values k or $\sigma + k$ for some $k \in \mathbb{Z}$. We represent this in fig. 2.

We remark that the set where $\alpha_\sigma = 1/2$ is the intersection of three rectangles and Ω:

$$\underbrace{\Omega \cap [-2+\sigma, -1] \times [-3+\sigma, -2]}_{\Omega_{11}} \cup \underbrace{\Omega \cap [-1+\sigma, 0] \times [-2, -1+\sigma]}_{\Omega_{12}} \cup \underbrace{\Omega \cap [\sigma, 1] \times [-1, 0]}_{\Omega_{13}},$$

while the set where $\alpha_\sigma = -1/2$ is the intersection of three rectangles and Ω:

$$\underbrace{\Omega \cap [-2, -2+\sigma] \times [-3, -2+\sigma]}_{\Omega_{-11}} \cup \underbrace{\Omega \cap [-1, -1+\sigma] \times [-3+\sigma, -1]}_{\Omega_{-12}} \cup \underbrace{\Omega \cap [0,\sigma] \times [-2+\sigma, 0]}_{\Omega{-13}}.$$

In other word, with the notations above,

$$K_t(s_1,s_2) = \frac{1}{2} \int_{\Omega_{11} \cup \Omega_{12} \cup \Omega_{13}} \phi'(t_1 + \tau)\,\mathrm{d}t_1\,\mathrm{d}t_2 - \frac{1}{2} \int_{\Omega_{-11} \cup \Omega_{-12} \cup \Omega_{-13}} \phi'(t_1 + \tau)\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

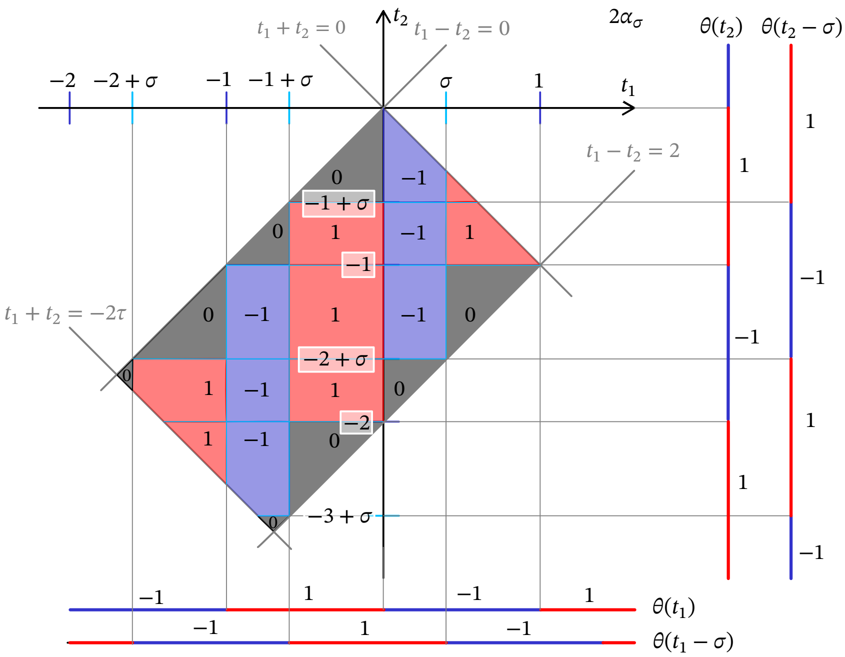

Figure 2: In light blue, the potential threshold for t_1 and t_2 where α_σ might change value. On the right, the values of $\theta(t_2 - \sigma)$ and $\theta(t_2)$. At the bottom, the values of $\theta(t_1 - \sigma)$ and $\theta(t_1)$. The diagonally placed rectangle is Ω. Inside Ω, we write what is the value of $2\alpha_\sigma(t_1, t_2)$.

Using Green's theorem, we get

$$K_t(s_1, s_2) = \frac{1}{2}\sum_{i=1}^{3}\oint_{\partial\Omega_{1i}} \phi(t_1+\tau)\,\mathrm{d}t_2 - \frac{1}{2}\sum_{i=1}^{3}\oint_{\partial\Omega_{-1i}} \phi(t_1+\tau)\,\mathrm{d}t_2.$$

The only thing left to do is identify the different cases where the $\Omega_{i,j}$ are empty, triangles, some other 4-polygon or 5-polygon and compute each of these integrals.

We detail one case, and give the result for the other with just a figure as explanation.

Step 1: Case $2t - 1 < s_1 + s_2 < 2t$ *(fig. 3).* In this case, the domains $\Omega_{i,j}$ look like the one of fig. 3. We have:

$$\begin{aligned}
2K_t(s_1,s_2) &= \underbrace{\int_0^\sigma \phi(s+\tau)\,\mathrm{d}s - \int_0^\sigma \phi(s+\tau)\,\mathrm{d}s}_{\text{“Diagonal” part of } \int_{\partial\Omega_{-1,3}}} + \underbrace{2\tau\phi(\tau) - 2\tau\phi(\sigma+\tau)}_{\text{“Vertical” part of } \int_{\partial\Omega_{-1,3}}} \\
&\quad \underbrace{- \int_\sigma^{-2\tau+1} \phi(s+\tau)\,\mathrm{d}s + \int_\sigma^{1} \phi(s+\tau)\,\mathrm{d}s}_{\text{“Diagonal” part of } \int_{\partial\Omega_{13}}} - \underbrace{2\tau\phi(\sigma+\tau)}_{\text{“Vertical” part of } \int_{\partial\Omega_{13}}} \\
&= \int_{-2\tau+1}^{1} \phi(s+\tau)\,\mathrm{d}s + 2\tau\phi(\tau) - 4\tau\phi(\sigma+\tau) \\
&= \int_{-2t+2s_2}^{0} \phi(s+t-s_2)\,\mathrm{d}s + 2(t-s_2)\phi(t-s_2) - 4(t-s_2)\phi(t-s_1)
\end{aligned}$$

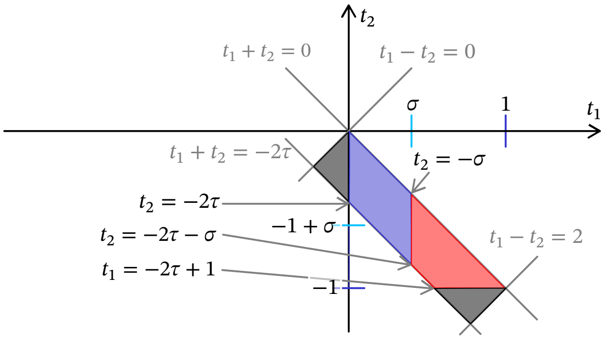

Figure 3: The equivalent of fig. 2 when $2t - 1 < s_1 + s_2$.

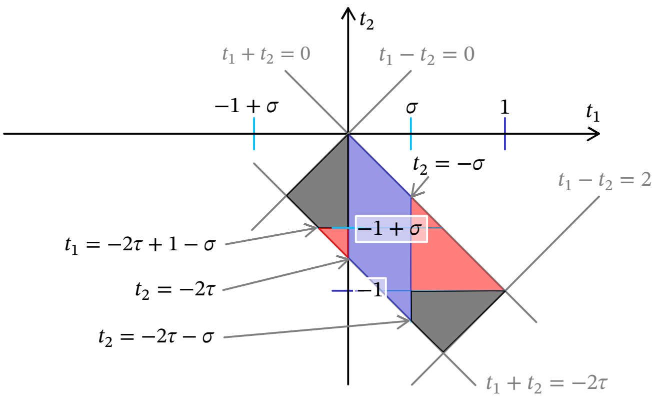

Figure 4: The equivalent of fig. 2 when $s_1 + s_2 < 2t - 1 < 2s_1 + 1$.

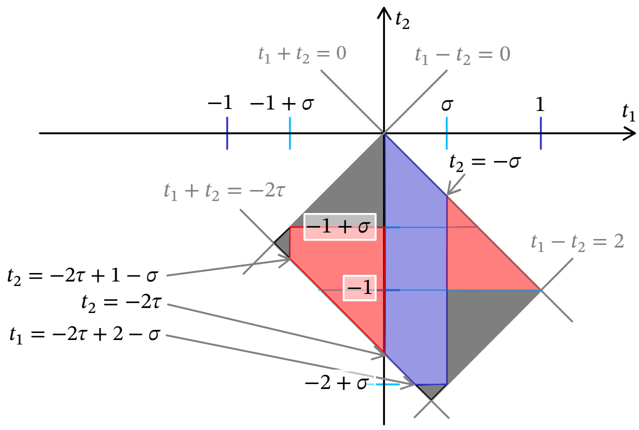

Figure 5: The equivalent of fig. 2 when $2s_1 < 2t - 2 < s_1 + s_2$.

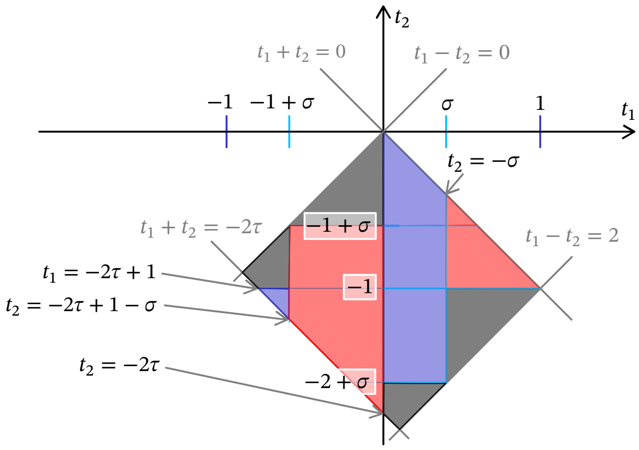

Figure 6: The equivalent of fig. 2 when $s_1 + s_2 < 2t - 2 < 2s_2$.

Step 2: Case $s_1 + s_2 < 2t - 1 < 2s_1 + 1$ (fig. 4).

$$2K_t(s_1, s_2) = \int_{s_2 - s_1}^{2-2t+s_2+s_1} \phi(s - s_2 + t)\,\mathrm{d}s + (4t - 1 - 3s_2 - s_1)\phi(t - s_2) - (1 + 2t - 3s_2 + s_1)\phi(t - s_1).$$

Step 3: Case $2s_1 < 2t - 2 < s_1 + s_2$ (fig. 5).

$$2K_t(s_1, s_2) = -\int_{2-2t+s_2+s_1}^{s_2 - s_1} \phi(s - s_2 + t)\,\mathrm{d}s + (4t - 1 - 3s_2 - s_1)\phi(t - s_2) - (1 + 2t - 3s_2 + s_1)\phi(t - s_1).$$

Step 4: Case $s_1 + s_2 < 2t - 2 < 2s_2$ (fig. 6).

$$2K_t(s_1, s_2) = -\int_{0}^{2-2t+2s_2} \phi(s + t - s_2)\,\mathrm{d}s + (1 + 2t - 2s_2)\phi(t - s_2) - (-1 + 4t - 4s_2)\phi(t - s_1).$$

Step 5: Case $2s_2 - 1 < 2t - 3 < s_1 + s_2$ (fig. 7).

$$2K_t(s_1, s_2) = \int_{2-2t+2s_2}^{0} \phi(s - s_2 + t)\,\mathrm{d}s + (1 + 2t - 2s_2)\phi(t - s_2) - (-1 + 4t - 4s_2)\phi(t - s_1).$$

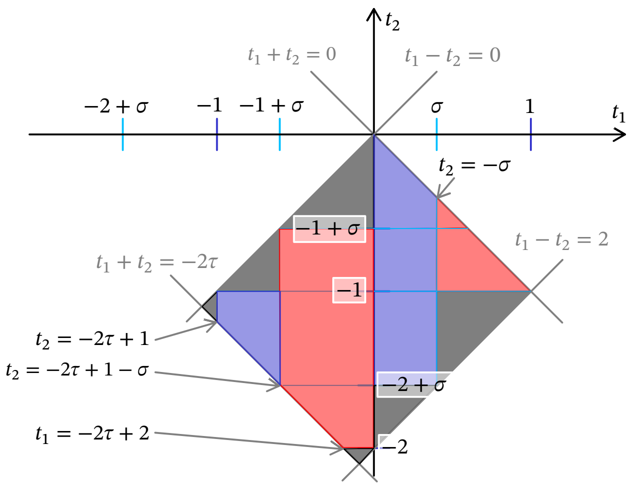

Figure 7: The equivalent of fig. 2 when $2s_2 - 1 < 2t - 3 < s_1 + s_2$.

Step 6: Case $s_1 + s_2 < 2t - 3 < 2s_1 + 1$ (fig. 8).

$$2K_t(s_1, s_2) = \int_{s_2-s_1}^{4-2t+s_2+s_1} \phi(s + t - s_2)\,\mathrm{d}s + (-2 + 4t - 3s_2 - s_1)\phi(t - s_2) - (2 + 2t - 3s_2 + s_1)\phi(t - s_1).$$

Step 7: Case $2s_1 < 2t - 4 < s_1 + s_2$ (fig. 9).

$$2K_t(s_1, s_2) = \int_{s_2-s_1}^{4-2t+s_2+s_1} \phi(s + t - s_2)\,\mathrm{d}s + (-2 + 4t - 3s_2 - s_1)\phi(t - s_2) - (2 + 2t - 3s_2 + s_1)\phi(t - s_1). \qquad \square$$

When the control u steers the linearized equation (8) from 0 to 0, we can prove that this kernel acts as another, simpler one.

Proposition 17. *Let $T \in (1, 2)$. Let $\phi \in C^1(\mathbb{T})$ that is 1-periodic. We define the* reduced kernel *$K_T^{\mathrm{red}} : [0, T-1]^2 \to \mathbb{R}$ by*

$$K_T^{\mathrm{red}}(s_1, s_2) := \frac{3}{2}(1 - |s_2 - s_1|)\big(\phi(T - s_1 \vee s_2) - \phi(T - s_1 \wedge s_2)\big).$$

Let $u \in L^2(0, T)$ that steers the linearized equation (16) from 0 to 0 (i.e., $\zeta_1(u, T, \cdot) = 0$). Let $\zeta_2(u, \cdot, \cdot)$ be the second-order correction for the water-tank system, i.e., the solution of (23). Then,

$$(\zeta_2(u, T, \cdot), \phi)_{L^2(\mathbb{T})} = \int_{[0,T-1]^2} K_T^{\mathrm{red}}(s_1, s_2) u(s_1) u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2. \tag{28}$$

The two important points of this formula, is that the expression of the reduced kernel is simpler, and that we integrate on $[0, T-1]^2$ instead of $[0, T]^2$.

Proof. Step 1: Expression of K_T^{red} as a function of K_T. According to proposition 11, we have for every $T - 1 < s < 1$, $u(s) = 0$ and $u(s + 1) = u(s)$. Thus, according the proposition 16 we have

$$\begin{aligned}(\zeta_2(T, \cdot), \phi) &= \int_{[0,T]^2} K_T(s_1, s_2) u(s_1) u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &= \int_{[0,T-1]^2} \big(K_T(s_1, s_2) + K_T(1 + s_1, s_2) + K_T(s_1, 1 + s_2) + K_T(1 + s_1, 1 + s_2)\big) u(s_1) u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2.\end{aligned}$$

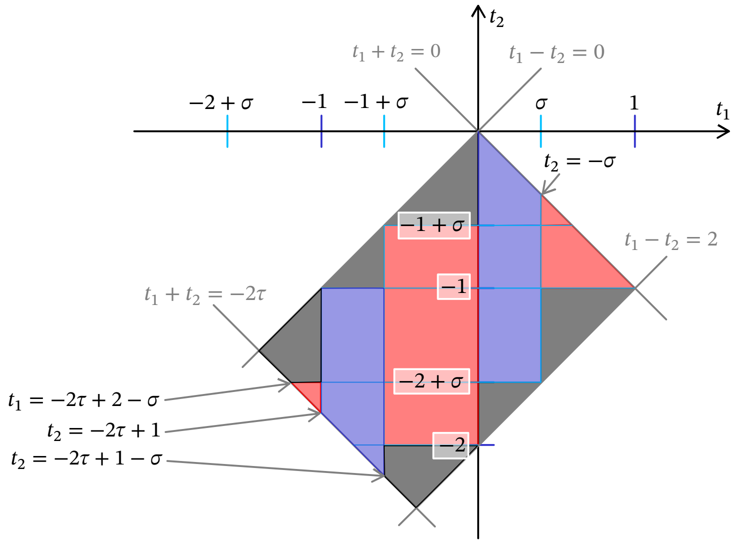

Figure 8: The equivalent of fig. 2 when $s_1 + s_2 < 2t - 3 < 2s_1 + 1$.

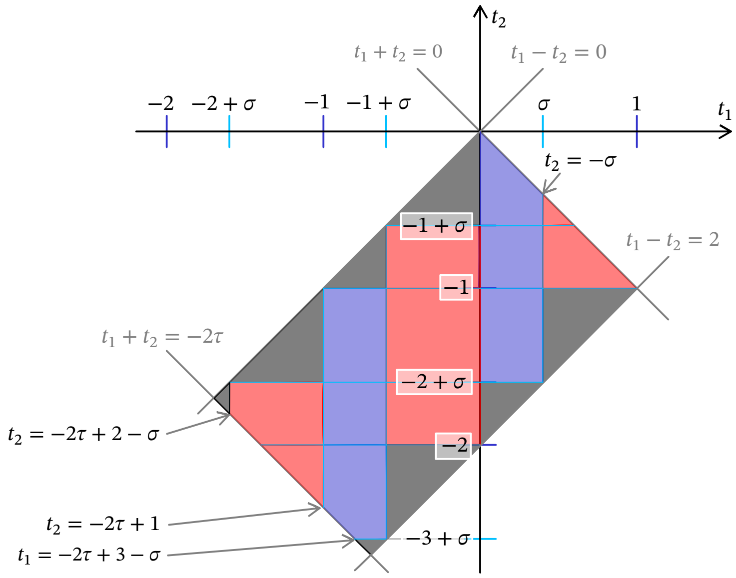

Figure 9: The equivalent of fig. 2 when $2s_1 < 2t - 4 < s_1 + s_2$.

Thus, equation (28) holds with $K_T^{\text{red}}(s_1,s_2) = K_T(s_1,s_2)+K_T(1+s_1,s_2)+K_T(s_1,1+s_2)+K_T(1+s_1,1+s_2)$. Since K_T (and also K_T^{red}) are symmetric in s_1, s_2, we may assume that $s_1 \le s_2$. Then, with $s_2' := 1 + s_2$ and $s_1' := s_1$, we have $s_1'+1 \le s_2'$, thus, according to proposition 16, we have $K_T(s_1', s_2') = -K_T(1+s_1', s_2')$. Thus, $K_T(s_1, 1+s_2) + K_T(1+s_1, 1+s_2) = 0$ and $K_T^{\text{red}}(s_1, s_2) = K_T(s_1,s_2) + K_T(1+s_1, s_2)$.

We end the computation by using the formula for K_T of proposition 16. We have $0 < s_1 \le s_2 < T-1$ and $1 < T < 2$. So $2T-4 < 0 < s_1+s_2 < 2T-2$. We consider two cases: $2T-3 < s_1+s_2 < 2T-2$ and $2T-4 < s_1+s_2 < 2T-3$.

Step 2: Case $2T-3 < s_1+s_2 < 2T-2$. To compute $K_T(s_1,s_2)$, we use the third case of the expression (27) of K_T. To compute $K_T(1+s_1,s_2)$, we remark that with $s_1' := s_2$ and $s_2' := 1+s_1$, we have $s_1' < s_2'$ and $2T-2 < s_1'+s_2' < 2T-1$. Thus, $K_T(1+s_1,s_2) = K_T(s_1',s_2')$ is computed with the second case of the expression (27) of K_T. We get

$$\begin{aligned}
2K_T^{\text{red}}(s_1,s_2) &= 2K_T(s_1,s_2) + 2K_T(s_1',s_2') \\
&= \int_{2-2t+2s_2}^{0} \phi(s+t-s_2)\,\mathrm{d}s + (1+2t-2s_2)\phi(t-s_2) - (-1+4t-4s_2)\phi(t-s_1) \\
&\quad + \int_{s_2'-s_1'}^{2-2t+s_2'+s_1'} \phi(s+t-s_2')\,\mathrm{d}s + (-1+4t-3s_2'-s_1')\phi(t-s_2') - (1+2t-3s_2'+s_1')\phi(t-s_1') \\
&= \int_{2-2t+2s_2}^{0} \phi(s+t-s_2)\,\mathrm{d}s + (1+2t-2s_2)\phi(t-s_2) - (-1+4t-4s_2)\phi(t-s_1) \\
&\quad + \int_{1+s_1-s_2}^{3-2t+s_1+s_2} \phi(s+t-s_1)\,\mathrm{d}s + (-4+4t-3s_1-s_2)\phi(t-s_1) - (-2+2t-3s_1+s_2)\phi(t-s_2) \\
&= \int_{2-2t+2s_2}^{0} \phi(s+t-s_2)\,\mathrm{d}s + \int_{1+s_1-s_2}^{3-2t+s_1+s_2} \phi(s+t-s_1)\,\mathrm{d}s \\
&\quad + (3-3s_2+3s_1)\phi(t-s_2) - (3-3s_2+3s_1)\phi(t-s_1).
\end{aligned}$$

In the second integral, we make the change of variables $s' = s + s_2 - s_1$:

$$\begin{aligned}
2K_T^{\text{red}}(s_1,s_2) &= \int_{2-2t+2s_2}^{0} \phi(s+t-s_2)\,\mathrm{d}s + \int_{1}^{3-2t+2s_2} \phi(s+t-s_2)\,\mathrm{d}s + 3(1-s_2+s_1)\big(\phi(t-s_2)-\phi(t-s_1)\big) \\
&= 3(1-s_2+s_1)\big(\phi(t-s_2)-\phi(t-s_1)\big),
\end{aligned}$$

where we used the 1-periodicity of ϕ to cancel the two integrals. Since we swapped s_1 and s_2 to have $s_1 = s_1 \vee s_2$ and $s_2 = s_1 \wedge s_2$, this is indeed the claimed formula.

Step 3: Case $2T-4 < s_1+s_2 < 2T-3$. This case is treated in the same way, the only difference being that $K_T(s_1,s_2)$ is computed using the fourth case of the expression (27) of K_T, and $K_T(1+s_1,s_2)$ is computed using the third case of the same expression. We get the same formula. □

3.3 Coercivity of the kernel

In this section, we use the expression of $(\zeta_2(u,T,\cdot),\phi)$ given in proposition 17 to prove that when $1 < T < 2$, $|\zeta_2(u,T,\cdot)|$ is lower-bounded by essentially $\|u\|_{H^{-1}}^2$. To do that, we first have to choose the right function ϕ.

Definition 18. Let $1 < T < 2$ and let ϕ be a C^∞ 1-periodic function such that $\phi(s) = s$ in $[1, T]$.

Proposition 19. *If $1 < T < 2$ and $u \in L^2(0,T)$ steers a solution of the linearized equation (16) from 0 to 0 (i.e., $\zeta_1(u,T,\cdot) = 0$) and if $\int_0^T u(t)\,\mathrm{d}t = 0$, then denoting $U(t) = \int_0^t u(s)\,\mathrm{d}s$,*

$$(\zeta_2(u,T,\cdot),\phi)_{L^2} \geq 3(2-T)\|U\|^2_{L^2(0,T-1)},$$

where ϕ is a function given in definition 18.

This proposition uses the following computation:

Lemma 20. *Let $I = (a,b)$ with $a < b$, and let $K \in H^1(I^2) \cap H^2(I^2 \setminus \{s_1 = s_2\})$. Let $R \in L^2(I^2)$ such that for $s_1 \neq s_2$, $R(s_1,s_2) = \partial_{s_1,s_2}K(s_1,s_2)$, let $w(s) := \partial_{s_1}K(s,s+0) - \partial_{s_1}K(s,s-0)$, and let $g(s) := \partial_{s_1}K(s,b) + \partial_{s_2}K(b,s)$. Then, for every $u \in L^2(a,b)$, with $U(t) := \int_a^t u(s)\,\mathrm{d}s$, we have*

$$\int_{I^2} K(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 = \int_I w(s)|U(s)|^2\,\mathrm{d}s + \int_{I^2} R(s_1,s_2)U(s_1)U(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$
$$- U(b)\int_I g(s)U(s)\,\mathrm{d}s + K(b,b)U(b)^2.$$

Proof. The proof formally consists in integrating by parts in s_1 and s_2. The first integration by parts is justified:

$$\int_{I^2} K(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 = -\int_{I^2} \partial_{s_1}K(s_1,s_2)U(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 + \int_I K(b,s_2)U(b)u(s_2)\,\mathrm{d}s_2. \tag{29}$$

Now we split the integral in two parts: $s_2 < s_1$ and $s_1 < s_2$:

$$\begin{aligned}
\int_{I^2} \partial_{s_1}K(s_1,s_2)U(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 &= \int_I\Big(\int_a^{s_1} \partial_{s_1}K(s_1,s_2)u(s_2)\,\mathrm{d}s_2 + \int_{s_1}^b \partial_{s_1}K(s_1,s_2)u(s_2)\,\mathrm{d}s_2\Big)U(s_1)\,\mathrm{d}s_1\\
&= \int_I\Big(-\int_a^{s_1} \partial_{s_1,s_2}K(s_1,s_2)U(s_2)\,\mathrm{d}s_2 + \partial_{s_1}K(s_1,s_1-0)U(s_1)\\
&\quad - \int_{s_1}^b \partial_{s_1,s_2}K(s_1,s_2)U(s_2)\,\mathrm{d}s_2\\
&\quad + \partial_{s_1}K(s_1,b)U(b) - \partial_{s_1}K(s_1,s_1+0)U(s_1)\Big)U(s_1)\,\mathrm{d}s_1\\
&= -\int_{I^2} R(s_1,s_2)U(s_1)U(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2\\
&\quad - \int_I w(s)U(s)^2\,\mathrm{d}s + U(b)\int_I \partial_{s_1}K(s_1,b)U(s_1)\,\mathrm{d}s_1.
\end{aligned}$$

Moreover,

$$\int_I K(b,s_2)U(b)u(s_2)\,\mathrm{d}s_2 = -\int_I \partial_{s_2}K(b,s_2)U(s_2)\,\mathrm{d}s_2 + K(b,b)U(b).$$

Plugging these two formulas into eq. (29) proves the lemma. □

Proof of proposition 19. We first simplify the expression of K_T^{red} given by proposition 17. For $0 < s_1, s_2 < T-1$, we have $1 < T - s_1 \vee s_2 \leq T - s_1 \wedge s_2 < T$, thus, according to the definition of ϕ, we have for $0 < s_1, s_2 < T-1$,

$$\begin{aligned} K_T^{\mathrm{red}}(s_1,s_2) &= \frac{3}{2}(1-|s_2-s_1|)\big((T-s_1\vee s_2)-(T-s_1\wedge s_2)\big) \\ &= -\frac{3}{2}(1-|s_2-s_1|)|s_2-s_1| \\ &= \frac{3}{2}(-|s_2-s_1|+(s_2-s_1)^2). \end{aligned}$$

Thus, according to proposition 17, if u is as in the statement of proposition 19,

$$(\zeta_2(u,T,\cdot),\phi) = -\frac{3}{2}\int_{[0,T-1]^2}|s_2-s_1|u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 + \frac{3}{2}\int_{[0,T-1]^2}(s_2-s_1)^2u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \tag{30}$$

With the notations of lemma 20 with $K = K_T^{\mathrm{red}}$, we have

$$R(s_1,s_2) = -3 \quad w(s) = 3.$$

Moreover, since $\int_0^T u(t)\,\mathrm{d}t = 0$, according to proposition 11, we have $\int_0^T u(t)\,\mathrm{d}t = 2\int_0^{T-1} u(t)\,\mathrm{d}t = 0$, hence the boundary term $U(T)$ is zero. Plugging the formula of lemma 20 into the expression (30), we get

$$(\zeta_2(u,T,\cdot),\phi) = 3\|U\|^2_{L^2(0,T-1)} - 3\left(\int_0^{T-1} U(s)\,\mathrm{d}s\right)^2. \tag{31}$$

According to Cauchy-Schwarz inequality, we have $|\int_0^{T-1} U(s)\,\mathrm{d}s| \leq \sqrt{T-1}\|U\|_{L^2(0,T-1)}$. Thus

$$(\zeta_2(u,T,\cdot),\phi) \geq 3(2-T)\|U\|^2_{L^2(0,T-1)}.$$

□

Remark 21. How did we choose the ϕ of definition 18? It turns out that if ϕ is monotone on $[1,T]$, the assertion

$$\int_0^{T-1} u(t)\,\mathrm{d}t = 0 \implies \int_{[0,T-1]^2} K_T^{\mathrm{red}}(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \geq c\|U\|^2_{L^2(0,T-1)}$$

is equivalent to the condition $\int_1^T \phi'(s)\,\mathrm{d}s \int_1^T (\phi'(s))^{-1}\,\mathrm{d}s < (3-T)^2$ (we sketch the proof of this fact in appendix A). Hence, the smaller the left-hand-side of this condition, the larger the time of non-local-controllability. With some calculus of variations, we can see that if ϕ minimizes the left-hand side, then ϕ' is constant on $[1,T]$, hence our choice of ϕ.

Remark 22. The hypothesis that $\int_0^T u(t)\,\mathrm{d}t = 0$ in proposition 19 is essential. Indeed, K_T^{red} is continuous and $K_T^{\mathrm{red}}(s,s) = 0$. Hence, if we chose a sequence $(u_n)_{n\in\mathbb{N}}$ of $L^2(0,T-1)$ that converges in measure to δ_{t_0} for some fixed $t_0 \in (0,T-1)$, we get

$$\int_{[0,T-1]^2} K_T^{\mathrm{red}}(s_1,s_2)u_n(s_1)u_n(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \xrightarrow[n\to+\infty]{} K_T^{\mathrm{red}}(t_0,t_0) = 0.$$

Moreover, we have

$$U_n(t) := \int_0^t u_n(s)\,\mathrm{d}s \xrightarrow[n\to+\infty]{} \begin{cases} 0 & \text{if } t < t_0, \\ 1 & \text{if } t > t_0, \end{cases}$$

hence $\|U_n\|_{L^2(0,T-1)} \xrightarrow[n\to+\infty]{} \sqrt{T-t_0} > 0$. This proves that the quadratic map

$$u \mapsto \int_{[0,T-1]^2} K^{\mathrm{red}}(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

has no "H^{-1}-coercivity".

4 Nonlinear equation

The proposition 19 shows that if the time T is smaller than 2 and if u steers the linearized equation (16) from 0 to 0, then $\|\zeta_2(u, T, \cdot)\|_{L^2} \geq c\|U\|^2_{L^2(0,T)}$ (where $U(t) = \int_0^t u(s)\,\mathrm{d}s$). As in the previous section, we fix $T \in (1, 2)$. Our aim now is to prove that the solution of the *nonlinear* equation also have this property, as long as $\|u\|_{C^0}$ is small enough. As a consequence, one cannot move the water-tank in time T with a control small in C^0-norm, and that finishes the proof of theorem 1.

To this end, we use the fact that if $\|u\|$ is small enough, the solution of the nonlinear equation is well approximated by $(h_1, v_1) + (h_2, v_2)$, where (h_1, v_1) solves the linearized system (8) and (h_2, v_2) solves the second order system (9).

4.1 Well-posedness of the water-tank system

In this section, we state several basic results on the nonlinear system related to the water-tank system (1). We begin with the well-posedness of the water-tank system, where, as in the rest of the article, $g = 1$ and $L = 1$.

Proposition 23. *Let $T > 0$. There exists $\epsilon > 0$ such that for $(H_0, v_0) \in [C^1([0,1])]^2$ that satisfies*

$$\|u\|_{C^0([0,T])} + \|(H_0, v_0) - (1,0)\|_{C^1([0,1])} < \epsilon,$$

as well as the compatibility conditions

$$\partial_x H_0(0) = \partial_x H_0(1) = -u(0),$$

there exists a unique solution $(H_{\mathrm{nl}}, v_{\mathrm{nl}}) \in [C^1([0,T] \times [0,1])]^2$ of the water-tank system (1) with $H_{\mathrm{nl}}(0,x) = H_0(x)$ and $v_{\mathrm{nl}}(0,x) = v_0(x)$. Moreover,

$$\|(H_{\mathrm{nl}}, v_{\mathrm{nl}}) - (1,0)\|_{C^1([0,T]\times[0,1])} \leq C\Big(\|u\|_{C^0([0,T])} + \|(H_0, v_0) - (1,0)\|_{C^1([0,1])}\Big), \tag{32}$$

for some positive constant C depending only on T.

Proof. In this proof, we drop the index nl and write just (H, v) for $(H_{\mathrm{nl}}, v_{\mathrm{nl}})$.

Standard results for the well-posedness of hyperbolic systems assume that all coefficients are at least C^1, but here we assume that u is only C^0. In order to achieve that, we note that if (H, v) solves the water tank system (1), then with V defined by $v(t,x) = V(t,x) - U(t)$, where, as usual, $U(t) = \int_0^t u(s)\,\mathrm{d}s$, the water-tank system becomes

$$\begin{cases} \partial_t H + \partial_x((V - U)H) = 0; \\ \partial_t V + \partial_x \left(H + \frac{(V-U)^2}{2}\right) = 0; \\ V(t,0) = V(t,1) = U(t), \end{cases} \tag{33}$$

where all the coefficients are now C^1. This system can be written in the form

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \begin{pmatrix} V-U & H \\ 1 & V-U \end{pmatrix} \partial_x \begin{pmatrix} H \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } [0,T] \times [0,1] \tag{34}$$

and

$$V(t,0) = V(t,1) = U(t) \text{ in } [0,T]. \tag{35}$$

System (34) and (35) is not in the standard form of the quasilinear hyperbolic system since the control U also appears in the nonlinearity. Nevertheless, the proof can be derived from the standard fixed point arguments, see, e.g., [21, Chapter 4] and [16, The proof of Lemma 2.2]. We now outline the proof. Set

$$(H^{(0)}, V^{(0)})(t,x) = (H_0(x), V_0(x)) \quad \text{in } [0,T]\times[0,1],$$

and define $(H^{(n)}, V^{(n)})$ in $[0,T]\times[0,L]$ for $n \geq 1$ by

$$\partial_t \begin{pmatrix} H^{(n)} \\ V^{(n)} \end{pmatrix} + \begin{pmatrix} V^{(n-1)} - U & H^{(n-1)} \\ 1 & V^{(n-1)} - U \end{pmatrix} \partial_x \begin{pmatrix} H^{(n)} \\ V^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } [0,T]\times[0,1], \tag{36}$$

with the corresponding boundary conditions. Using the characteristic method, we have, if

$$\|(H^{(n-1)} - 1, V^{(n-1)}, U)\|_{C^1([0,T]\times[0,1])} \leq C\epsilon,$$

then

$$\|(H^{(n)} - 1, V^{(n)})\|_{C^0([0,T]\times[0,1])} \leq C\Big(\|U\|_{C^0([0,T])} + \|(H_0, V_0) - (1,0)\|_{C^0([0,1])}\Big)$$

and by taking the derivative of the equation with respect to t, we also obtain

$$\|(H^{(n)} - 1, V^{(n)})\|_{C^1([0,T]\times[0,1])} \leq C\Big(\|u\|_{C^0([0,T])} + \|(H_0, V_0) - (1,0)\|_{C^1([0,1])}\Big).$$

We derive that

$$\|(H^{(n)} - 1, V^{(n)})\|_{C^1([0,T]\times[0,1])} \leq C\epsilon.$$

Set

$$\rho_n(r) := \sup_{\substack{|(t,x)-(s,y)|<r \\ (t,x),(s,y)\in[0,T]\times[0,1]}} |(H^{(n)}(t,x) - H^{(n)}(s,y), V^{(n)}(t,x) - V^{(n)}(s,y))|$$

Using the characteristic, we can prove that there exists a positive constant γ depending only on T such that for ϵ sufficiently small,

$$\rho_n(r) \leq C\Big(\sup_{\substack{|t-s|<\gamma r \\ t,s\in[0,T]}} |u(t) - u(s)| + \sup_{\substack{|x-y|<\gamma r \\ x,y\in[0,L]}} |(H_0, V_0) - (1,0)|\Big).$$

Using Ascoli's theorem, one can conclude that there exists up to a subsequence $(H^{(n)}, V^{(n)})$ converges to (H, V) in $C^1([0,T]\times[0,1])$.

Considering the system solved by $(H^{(n+1)} - H^{(n)}, V^{(n+1)} - V^{(n)})$, one can check that

$$\|(H^{(n+1)} - H^{(n)}, V^{(n+1)} - V^{(n)})\|_{C^0([0,T]\times[0,1])} \leq C\epsilon\|(H^{(n)} - H^{(n-1)}, V^{(n)} - V^{(n-1)})\|_{C^0([0,T]\times[0,1])}. \tag{37}$$

Thus $(H^{(n)}, V^{(n)})$ converges to (H, V) in $C^0([0,T]\times[0,L])$. We thus derive that $(H, V) \in C^1([0,T]\times[0,1])$ is the corresponding solution.

The uniqueness follows as in (37). □

Remark 24. We do not need this for the proofs below, but it is worth noting that standard methods using the propagation along characteristics can be used to prove the lack of local-controllability around equilibrium states in time $T < T_*$. Let us sketch it. Consider the characteristic speeds $\lambda_\pm$ and Riemann invariants $R_\pm$, which are given by[2]

$$\lambda_\pm = v \pm \sqrt{H};$$
$$R_\pm = v \pm 2\sqrt{H} + U.$$

[2]The Riemann invariant as defined in [2, Section 1.4] do not have the $+U$ term. But in our case, it is convenient to add it.

We have
$$\begin{cases}(\partial_t+\lambda_\pm\partial_x)R_\pm=0;\\ R_\pm(t,0)=-R_\mp(t,1)+2U.\end{cases}$$
Consider also the characteristics, i.e., the solutions $x_\pm$ of the Cauchy problem
$$\begin{cases}\partial_t x_\pm(t,t_0,x_{t_0})=\lambda_\pm(x_\pm(t,t_0,x_{t_0}));\\ x_\pm(t_0,t_0,x_{t_0})=x_{t_0}.\end{cases}$$
Then, differentiating in t and using the equation for R_+, we get that $R_+(t,x_+(t,t_0,0))$ does not depend on t (as long as $x_+(t,t_0,0)$ is defined, i.e., stays inside $[0,1]$). Hence
$$R_+(t,x(t,t_0,0))=R_+(t_0,0)=-R_-(t_0,1)+2U(t_0).$$
Hence, if $R_\pm(T,\cdot)=0$, $0<t_0<T$, and if $x_+(T,t_0,0)$ is defined, $U(t_0)=0$. The characteristic speed depends on the solution, and thus on the control, but if the control is small, the characteristic speeds are $\lambda_\pm(t,x)=\pm1+O(\|u\|_{C^0})$, which implies that $x_+(t,t_0,0)=t-t_0+O(\|u\|_{C^0})$. Hence, the computations outlined above are valid if $T<1-O(\|u\|_{C^0})$.

4.2 Error estimates

In this section, $(H_{\mathrm{nl}}(u),v_{\mathrm{nl}}(u))=(1+h_{\mathrm{nl}}(u),v_{\mathrm{nl}}(u))$ is the solution of the water-tank system (1) with control u. We will often conflate this solution and $\zeta_{\mathrm{nl}}(u):=\mathcal{C}(h_{\mathrm{nl}}(u),v_{\mathrm{nl}}(u))$. The same will be done for the solution $(h_1(u),v_1(u))$ of the linearized system (8) and $\zeta_1(u):=\mathcal{C}(h_1(u),v_1(u))$ (solution of (16)), as well as the solution $(h_2(u),v_2(u))$ of (9) and $\zeta_2(u):=\mathcal{C}(h_2(u),v_2(u))$. If anything, this will make the notations more lightweight.

We will also set $w_{\mathrm{nl}}(u):=-\mathcal{C}(\partial_x(h_{\mathrm{nl}}(u)v_{\mathrm{nl}}(u)),\partial_x(v_{\mathrm{nl}}(u)^2/2))$, so that $\zeta_{\mathrm{nl}}(u)$ satisfies $(\partial_t+\partial_x)\zeta_{\mathrm{nl}}(u,t,x)=w_{\mathrm{nl}}(u,t,x)+u(t)\theta(x)$. We also denote the right-hand side of the equation (23) satisfied by $\zeta_2(u)$ by $w_1(u,t,x)$, i.e., $w_1(u)=-\mathcal{C}(\partial_x(h_1(u)v_1(u)),\partial_x(v_1(u)^2/2))$. Finally, we set $\delta_1(u):=\zeta_{\mathrm{nl}}(u)-\zeta_1(u)$ and $\delta_2(u):=\zeta_{\mathrm{nl}}(u)-\zeta_1(u)-\zeta_2(u)$.

In this subsection, we prove estimates on the following error terms:

- in lemma 26, an estimate on $\delta_2=\zeta_{\mathrm{nl}}-\zeta_1-\zeta_2$;
- in lemma 27, we bound $\zeta_2(\tilde{u},T,\cdot)-\zeta_2(u,T,\cdot)$.

The aim is to prove that these terms cannot counter the positivity of the term $3(2-T)\|U\|_{L^2}^2$ that appears in proposition 19.

We start with an estimate for the nonlinear equation, which is a consequence of the nonlinear well-posedness (proposition 23) and the linear estimates (proposition 9):

Corollary 25. *Let $T>0$. There exists $\eta>0$ and $C>0$ such that for every $u\in C^0([0,T])$ with $u(0)=0$ and $\|u\|_{C^0([0,T])}<\delta$, there exists a unique solution $(H_{\mathrm{nl}},v_{\mathrm{nl}})\in\left[C^1([0,T]\times[0,1])\right]^2$ of the water-tank system (1) with $(H_{\mathrm{nl}},v_{\mathrm{nl}})(0,\cdot)=(1,0)$. Moreover, with the notation ζ_{nl} defined at the beginning of this section, we have*
$$\|\zeta_{\mathrm{nl}}\|_{L^2_tL^2_x}\le C\|U\|_{L^2(0,T)}.$$

Proof. The existence and uniqueness is a consequence of the well-posedness (proposition 23). Let us now prove the inequality. We write $\zeta_{\mathrm{nl}}=\zeta_1+\delta_1$.

We have $(\partial_t+\partial_x)\zeta_1(u,t,x)=u(t)\theta(x)$ and $(\partial_t+\partial_x)\delta_1(u,t,x)=w_{\mathrm{nl}}(u,t,x)$. Hence, according to proposition 9, we have $\|\zeta_1\|_{L^2_tL^2_x}\le C\|U\|_{L^2}$ and $\|\delta_1\|_{L^2_tL^2_x}\le C\|w_{\mathrm{nl}}\|_{L^2}^2$. Since w_{nl} can be written as

$-\partial_x r_{\mathrm{nl}}$ where $r_{\mathrm{nl}}(t,x)$ is a quadratic form of $\zeta_{\mathrm{nl}}(t,x)$ and $\zeta_{\mathrm{nl}}(t,-x)$ (lemma 13), we have $\|w_{\mathrm{nl}}\|_{L^2_tL^2_x} \leq C\|\partial_x\zeta_{\mathrm{nl}}\|_{L^\infty}\|\zeta_{\mathrm{nl}}\|_{L^2_tL^2_x}$. Thus,

$$\|\zeta_{\mathrm{nl}}\|_{L^2_tL^2_x} \leq C\big(\|U\|_{L^2} + \|\partial_x\zeta_{\mathrm{nl}}\|_{L^\infty}\|\zeta_{\mathrm{nl}}\|_{L^2_tL^2_x}\big).$$

Finally, since $\|\partial_x\zeta_{\mathrm{nl}}\|_{L^\infty} \leq \|\mathcal{C}(h_{\mathrm{nl}}, v_{\mathrm{nl}})\|_{W^{1,\infty}} \leq 2\|(h_{\mathrm{nl}}, v_{\mathrm{nl}})\|_{C^1}$ (see remark 5), we have according to the well-posedness estimate of proposition 23 $\|\partial_x\zeta_{\mathrm{nl}}\|_{L^\infty} \leq C\|u\|_{C^0} \leq C\eta$. Thus,

$$\|\zeta_{\mathrm{nl}}\|_{L^2_tL^2_x} \leq C\|U\|_{L^2} + C\eta\|\zeta_{\mathrm{nl}}\|_{L^2_tL^2_x},$$

which implies for η small enough

$$\|\zeta_{\mathrm{nl}}\|_{L^2_tL^2_x} \leq \frac{C}{1-C\eta}\|U\|_{L^2}. \qquad \square$$

Next, we prove the approximation property:

Lemma 26. *Let $\phi \in C^1(\mathbb{T})$. Let $T > 0$ and $u \in H^1(0,T)$ with $u(0) = 0$ and $\|u\|_{H^1_0} < \eta$, and set $U(t) := \int_0^t u(s)\,\mathrm{d}s$. Then, with the notations above, for some $C > 0$ independent of u,*

$$\|\delta_1(u,\cdot,\cdot)\|_{L^\infty_tL^2_x} \leq C\|U\|_{L^2(0,T)}\|u\|_{C^0(0,T)}; \tag{38}$$

$$|(\delta_2(u,T,\cdot),\phi)| \leq C\|U\|^2_{L^2(0,)}\|u\|_{C^0(0,T)}. \tag{39}$$

Proof. Step 1: Estimate of δ_1 in L^2-norm. We have $(\partial_t + \partial_x)\delta_1 = w_{\mathrm{nl}}$, thus, using Duhamel's formula,

$$\|\delta_1\|_{L^\infty_tL^2_x} \leq C\|w_{\mathrm{nl}}\|_{L^1_tL^2_x}.$$

Since $w_{\mathrm{nl}} = -\mathcal{C}(\partial_x(h_{\mathrm{nl}}v_{\mathrm{nl}}), \partial_x(v_{\mathrm{nl}}^2/2))$, we can use lemma 13 to write $w_{\mathrm{nl}} = -\partial_x r_{\mathrm{nl}}$ with $r_{\mathrm{nl}}(t,x) = Q(\zeta_{\mathrm{nl}}(t,x), \zeta_{\mathrm{nl}}(t,-x))$. Thus,

$$\|\delta_1\|_{L^\infty_tL^2_x} \leq C\|\partial_x r_{\mathrm{nl}}\|_{L^2_tL^2_x}.$$

Since Q is a quadratic form (see lemma 13), $\partial_x r_{\mathrm{nl}}$ is a sum of products of ζ_{nl} and $\partial_x\zeta_{\mathrm{nl}}$ evaluated at (t,x) or $(t,-x)$. Thus, we get

$$\begin{aligned}\|\delta_1\|_{L^\infty_tL^2_x} &\leq C\|\zeta_{\mathrm{nl}}\|_{L^2_tL^2_x}\|\partial_x\zeta_{\mathrm{nl}}\|_{L^\infty_tL^\infty_x}\\ &\leq C\|\zeta_{\mathrm{nl}}\|_{L^2_tL^2_x}\|(h_{\mathrm{nl}}, v_{\mathrm{nl}})\|_{C^1([0,T]\times[0,1])},\end{aligned}$$

where we used that the change of variables $\mathcal{C}$ is such that for $(h,v) \in C^1([0,1])$ with $v(0) = v(1) = 0$, then $\|\mathcal{C}(h,v)\|_{W^{1,\infty}} \leq 2\|(h,v)\|_{C^1}$ (remark 5). Finally, using the well-posedness estimates of proposition 23 and corollary 25, we get

$$\|\delta_1\|_{L^\infty_tL^2_x} \leq C\|U\|_{L^2}\|u\|_{C^0}. \tag{40}$$

Step 2: Estimation on (δ_2, ϕ). The function δ_2 is solution of $(\partial_t + \partial_x)\delta_2 = w_{\mathrm{nl}} - w_1$. Thus, using the characteristics formula (lemma 6),

$$\begin{aligned}(\delta_2(u,T,\cdot),\phi) &= \int_{x\in\mathbb{T}} \delta_2(u,T,x)\phi(x)\,\mathrm{d}x\\ &= \int_{[0,T]\times\mathbb{T}} (w_{\mathrm{nl}} - w_1)(u,s,x+s-T)\phi(x)\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

We can use lemma 13 to write $w_{\mathrm{nl}} = -\partial_x r_{\mathrm{nl}}$ with $r_{\mathrm{nl}}(t,x) = Q(\zeta_{\mathrm{nl}}(t,x), \zeta_{\mathrm{nl}}(t,-x))$ and similarly for w_1. Thus, integrating by parts,

$$(\delta_2(u,T,\cdot),\phi) = \int_{[0,T]\times\mathbb{T}} (r_{\mathrm{nl}} - r_1)(u,s,x+s-T)\partial_x\phi(x)\,\mathrm{d}s\,\mathrm{d}x.$$

Thus,

$$|(\delta_2(u,T,\cdot),\phi)| \leq \|r_{\mathrm{nl}}(u) - r_1(u)\|_{L^1_t L^1_x}\|\phi(x)\|_{C^1}.$$

We recall that $r_{\mathrm{nl}}(t,x) = Q(\zeta_{\mathrm{nl}}(t,x), \zeta_{\mathrm{nl}}(t,-x))$ where Q is a quadratic form, and similarly for r_1. Thus, writing $aa' - bb' = \big((a-b)(a'+b') + (a'-b')(a+b)\big)/2$, we get

$$\begin{aligned}
|(\delta_2(u,T,\cdot),\phi)| \leq C\Big(&\|(\zeta_1 - \zeta_{\mathrm{nl}})(t,x)(\zeta_1(t,x) + \zeta_{\mathrm{nl}}(t,x))\|_{L^1_t L^1_x}\\
&+ \|(\zeta_1 - \zeta_{\mathrm{nl}})(t,-x)(\zeta_1(t,x) + \zeta_{\mathrm{nl}}(t,x))\|_{L^1_t L^1_x}\\
&+ \|(\zeta_1 - \zeta_{\mathrm{nl}})(t,x)(\zeta_1(t,-x) + \zeta_{\mathrm{nl}}(t,-x))\|_{L^1_t L^1_x}\\
&+ \|(\zeta_1 - \zeta_{\mathrm{nl}})(t,-x)(\zeta_1(t,-x) + \zeta_{\mathrm{nl}}(t,-x))\|_{L^1_t L^1_x}\Big)\\
\leq\ & C\|\zeta_1 - \zeta_{\mathrm{nl}}\|_{L^2_t L^2_x}(\|\zeta_1\|_{L^2_t L^2_x} + \|\zeta_{\mathrm{nl}}\|_{L^2_t L^2_x}).
\end{aligned}$$

Finally, using the estimate on δ_1 we obtained in the first step, the regularity estimate on ζ_1 of proposition 9 and the estimate on ζ_{nl} of corollary 25,

$$|(\delta_2(u,T,\cdot),\phi)| \leq C\|U\|_{L^2}\|u\|_{C^0}\|U\|_{L^2}.$$ □

We will also need to estimate $\zeta_2(u) - \zeta_2(\tilde{u})$.

Lemma 27. *Let $\phi \in C^1(\mathbb{T})$, $T > 0$ and $u, \tilde{u} \in L^2$. With the notations of lemma 26, and with $U(t) := \int_0^t u(s)\,\mathrm{d}s$ and $\widetilde{U}(t) := \int_0^t \tilde{u}(s)\,\mathrm{d}s$, for some $C > 0$ independent of $u, \tilde{u}$,*

$$|(\zeta_2(u,T,\cdot) - \zeta_2(\tilde{u},T\cdot),\phi)| \leq C\|U - \widetilde{U}\|_{L^2(0,T)}(\|U\|_{L^2(0,T)} + \|\widetilde{U}\|_{L^2(0,T)}).$$

Proof. We use the same notations w_1 and r_1 as lemma 13. The function $\zeta_2(u) - \zeta_2(\tilde{u})$ satisfies

$$(\partial_t + \partial_x)(\zeta_2(u) - \zeta_2(\tilde{u})) = w_1(u) - w_1(\tilde{u}).$$

Thus, according to the characteristics formula

$$(\zeta_2(u,T,\cdot) - \zeta_2(\tilde{u},T\cdot),\phi) = \int_{[0,T]\times\mathbb{T}} (w_1(u,s,x+s-T) - w_1(\tilde{u},s,x+s-T))\phi\,\mathrm{d}s\,\mathrm{d}x.$$

Since $w_1(u) = -\partial_x r_1(u)$, we integrate by parts to get

$$(\zeta_2(u,T,\cdot) - \zeta_2(\tilde{u},T\cdot),\phi) = \int_{[0,T]\times\mathbb{T}} (r_1(u,s,x+s-T) - r_1(\tilde{u},s,x+s-T))\partial_x\phi\,\mathrm{d}s\,\mathrm{d}x.$$

Thus,

$$|(\zeta_2(u,T,\cdot) - \zeta_2(\tilde{u},T\cdot),\phi)| \leq C\|r_1(u) - r_1(\tilde{u})\|_{L^1_t L^1_x}.$$

Recall that $r_1(u,t,x)$ is a linear combination of quadratic terms involving $\zeta_1(u,t,x)$ and $\zeta_1(u,t,x)$, (see lemma 13). Thus, writing $aa' - bb' = \big((a-b)(a'+b') + (a'-b')(a+b)\big)/2$, and using Hölder's inequality, we get

$$|(\zeta_2(u,T,\cdot) - \zeta_2(\tilde{u},T\cdot),\phi)| \leq C\|\zeta_1(u-\tilde{u})\|_{L^2_t L^2_x}(\|\zeta_1(u)\|_{L^2_t L^2_x} + \|\zeta_1(\tilde{u})\|_{L^2_t L^2_x}).$$

Finally, the regularity estimate for the linear equation (proposition 9) proves the theorem. □

4.3 Quadratic drift

We prove in this section a "quadratic drift" result. Theorem 1 follows easily from this result. We keep the notations $\zeta_{\mathrm{nl}}, \zeta_1, \delta_1$, etc. defined at the start of the previous subsection.

Lemma 28. *Let $\Pi : \zeta \in L^2(\mathbb{T}) \mapsto (\zeta - \zeta(\cdot + 1))/2$, which is the orthogonal projection on the reachable space for the linearized equation (remark 7 and lemma 10). Let $T \in (1, 2)$. There exist $\phi \in C^\infty(\mathbb{T})$, $c = c_T > 0$, and $\eta > 0$ such that for every $u \in C^0([0, T])$ with $u(0) = 0$ and $\|u\|_{C^0} < \eta$, if $\Pi\zeta_{\mathrm{nl}}(u, T, \cdot) = 0$ and $\int_0^T u(t)\,\mathrm{d}t = 0$,*

$$(\phi, \zeta_{\mathrm{nl}}(u, T, \cdot))_{L^2(\mathbb{T})} \geq c\|U\|^2_{L^2(0,T-1)},$$

where $U(t) := \int_0^t u(s)\,\mathrm{d}s$.

Proof. Let $T \in (1, 2)$. Let $\eta > 0$ such that lemma 26 holds. Reducing η if necessary, we may assume that $\eta < 1$. Let $u \in C^0(0, T)$ with $u(0) = 0$ and $\|u\|_{C^0} < \eta$ such that $\Pi\zeta_{\mathrm{nl}}(u, T, \cdot) = 0$.
Step 1: There exists a control $\tilde{u}$ close to u that steers the linearized equation from 0 to 0. We are looking for a control $\tilde{u}$ close to u such that $\zeta_1(u, T, \cdot) = 0$. We look for $\tilde{u}$ with the form $\tilde{u} = u + v$. The condition $\zeta_1(u + v, T, \cdot) = 0$ is equivalent to $\zeta_1(u, T, \cdot) = -\zeta_1(v, T, \cdot)$. Since $\Pi\zeta_{\mathrm{nl}}(u, T, \cdot) = 0$ by hypothesis and since $\Pi\zeta_1(u, T, \cdot) = \zeta_1(u, T, \cdot)$ (remark 7), we rewrite this as

$$\zeta_1(v, T, \cdot) = \Pi\delta_1(u, T, \cdot). \tag{41}$$

According to lemma 10, such a control v exists, and we can also choose it such that $\int_0^T v(t)\,\mathrm{d}t = 0$ and such that $\mathcal{V}(t) := \int_0^t v(s)\,\mathrm{d}s$ satisfies $\|\mathcal{V}\|_{L^2(0,T)} \leq C\|\Pi\delta_1(u, T, \cdot)\|_{L^2} \leq C\|\delta_1(u, T, \cdot)\|_{L^2}$. According to the estimate on δ_1 of lemma 26, this control is such that

$$\|\mathcal{V}\|_{L^2(0,T)} \leq C\|u\|_{C^0}\|U\|_{L^2(0,T)}. \tag{42}$$

Step 2: Estimating the difference $(\zeta_{\mathrm{nl}}(u, T, \cdot), \phi) - (\zeta_2(\tilde{u}, T, \cdot), \phi)$. Since $\zeta_1(u, T, \cdot)$ is 1-antiperiodic (remark 7), and since ϕ is 1-periodic, $(\zeta_1(u, T, \cdot), \phi) = 0$. Thus using the triangle inequality

$$\begin{aligned}\left|\left(\zeta_{\mathrm{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi\right)\right| &= \left|\left(\zeta_{\mathrm{nl}}(u, T, \cdot) - \zeta_1(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi\right)\right| \\ &\leq \left|\left(\zeta_{\mathrm{nl}}(u, T, \cdot) - \zeta_1(u, T, \cdot) - \zeta_2(u, T, \cdot), \phi\right)\right| + \left|\left(\zeta_2(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi\right)\right|\end{aligned}$$

The first term of the right-hand side is $|(\delta_2(u, T, \cdot), \phi)|$, and according to lemma 26, we have $|(\delta_2(u, T, \cdot), \phi)| \leq C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}$. According to lemma 27, the second term is bounded by $C\|\mathcal{V}\|_{L^2(0,T)}(\|U\|_{L^2(0,T)} + \|\mathcal{V}\|_{L^2(0,T)})$. Thus,

$$\left|\left(\zeta_{\mathrm{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi\right)\right| \leq C\|U\|^2_{L^2(0,T)}\|u\|_{C^0} + C\|\mathcal{V}\|_{L^2(0,T)}(\|\mathcal{V}\|_{L^2(0,T)} + \|U\|_{L^2(0,T)}).$$

Now, plugging the estimate on $\|\mathcal{V}\|_{L^2(0,T)}$ (eq. (42)), we get

$$\begin{aligned}\left|\left(\zeta_{\mathrm{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi\right)\right| &\leq C\|U\|^2_{L^2(0,T)}\|u\|_{C^0} + C\|U\|_{L^2(0,T)}\|u\|_{C^0}(\|U\|_{L^2(0,T)}\|u\|_{C^0} + \|U\|_{L^2(0,T)}) \\ &= C\|U\|^2_{L^2(0,T)}\|u\|_{C^0} + C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}(\|u\|_{C^0} + 1).\end{aligned}$$

Since we assumed that $\|u\|_{C^0} < \eta < 1$, we have

$$\left|\left(\zeta_{\mathrm{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi\right)\right| \leq C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}. \tag{43}$$

Step 3: Using the coercivity of the kernel. According to the estimate (43) from previous step and the inverse triangle inequality, we have

$$\begin{aligned}(\zeta_{\mathrm{nl}}(u, T, \cdot), \phi) &\geq (\zeta_2(\tilde{u}, T, \cdot), \phi) - \left|\left(\zeta_{\mathrm{nl}}(u, T, \cdot) - \zeta_2(\tilde{u}, T, \cdot), \phi\right)\right| \\ &\geq (\zeta_2(\tilde{u}, T, \cdot), \phi) - C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}\end{aligned}$$

Recall that $\zeta_1(\tilde{u}, T, \cdot) = 0$, and $\int_0^T u(t)\,\mathrm{d}T = 0$. Hence we can plug the the coercivity estimate of proposition 19, which gives us

$$(\zeta_{\mathrm{nl}}(u, T, \cdot), \phi) \geq 3(2 - T)\|\widetilde{U}\|^2_{L^2(0,T-1)} - C\|u\|_{C^0}\|U\|^2_{L^2(0,T)}, \tag{44}$$

where $\widetilde{U}(s) = \int_0^s \tilde{u}(s')\,\mathrm{d}s'$.

Step 4: Going back to U instead of $\widetilde{U}$. We now bound from below the term $\|\widetilde{U}\|^2_{L^2(0,T-1)}$. We have

$$2(U, \mathcal{V})_{L^2(0,T-1)} \leq \frac{1}{2}\|U\|^2_{L^2(0,T-1)} + 2\|\mathcal{V}\|^2_{L^2(0,T-1)}.$$

Since, $\tilde{u} = u + v$, this implies

$$\begin{aligned}\|\widetilde{U}\|^2_{L^2(0,T-1)} &= \|U\|^2_{L^2(0,T-1)} - 2(U, \mathcal{V})_{L^2(0,T-1)} + \|\mathcal{V}\|^2_{L^2(0,T-1)} \\ &\geq \frac{1}{2}\|U\|^2_{L^2(0,T-1)} - \|\mathcal{V}\|^2_{L^2(0,T-1)}.\end{aligned}$$

Using the bound on $\|\mathcal{V}\|_{L^2(0,T)}$ (eq. (42)), we get

$$\begin{aligned}\|\widetilde{U}\|^2_{L^2(0,T-1)} &\geq \frac{1}{2}\|U\|^2_{L^2(0,T-1)} - C\|u\|^2_{C^0}\|U\|^2_{L^2(0,T)}. \\ &\geq \frac{1}{2}\|U\|^2_{L^2(0,T-1)} - C\|u\|_{C^0}\|U\|^2_{L^2(0,T)}\end{aligned} \tag{45}$$

Plugging this into eq. (44), we get

$$(\zeta_{\mathrm{nl}}(u, T, \cdot), \phi) \geq \frac{3}{2}(2 - T)\|U\|^2_{L^2(0,T-1)} - C\|u\|_{C^0}\|U\|^2_{L^2(0,T)}. \tag{46}$$

Finally, we estimate $\|U\|^2_{L^2(0,T)}$ by $\|U\|^2_{L^2(0,T-1)}$. Since $\int_0^T u(s)\,\mathrm{d}s = 0$ and since since $u(t+1) = u(t)$ (proposition 11) we get $\int_0^{T-1} u(s)\,\mathrm{d}s = 0$. Thus, using again that $u(t+1) = u(t)$ for $0 < t < T-1$ and $u(t) = 0$ for $T-1 < t < 1$, we get

$$U(t) = \int_0^t u(s)\,\mathrm{d}s = \begin{cases} U(t) & \text{for } 0 < t < T-1 \\ \int_0^{T-1} u(s)\,\mathrm{d}s + \int_{T-1}^t 0\,\mathrm{d}s = 0 & \text{for } T-1 < t < 1 \\ \int_0^{T-1} u(s)\,\mathrm{d}s + \int_{T-1}^1 0\,\mathrm{d}s + \int_1^T u(s-1)\,\mathrm{d}s = U(t-1) & \text{for } 1 < t < T \end{cases}$$

Thus, $\|U\|^2_{L^2(0,T)} = 2\|U\|^2_{L^2(0,T-1)}$. Plugging this into eq. (46), we get

$$(\phi, \zeta_{\mathrm{nl}}(u, T, \cdot))_{L^2(\mathbb{T})} \geq \frac{3}{2}(2 - T)\|U\|^2_{L^2(0,T-1)} - C\|u\|_{C^0(0,T)}\|U\|^2_{L^2(0,T-1)}.$$

Step 5: Conclusion. Finally, since $\|u\|_{C^0(0,T)} < \eta$, we rewrite this as

$$(\phi, \zeta_{\mathrm{nl}}(u, T, \cdot))_{L^2(\mathbb{T})} \geq (\frac{3}{2}(2 - T) - C\eta)\|U\|^2_{L^2(0,T-1)}.$$

If η is small enough, this is the claimed lower bound. □

A On the positivity of a class of quadratic forms

In this appendix, we sketch the proof of the following proposition.

Proposition 29. *Let $I = [a, b]$ with $a < b$, let $\phi : I \to \mathbb{R}$ be C^1 and such that $\phi' \geq c > 0$ and ϕ' non constant, and let $\epsilon \in \mathbb{R}$. Set $K(s_1, s_2) := (1 + \epsilon|s_2 - s_1|)(\phi(s_1 \wedge s_2) - \phi(s_1 \vee s_2))$, and denote by Q_K the associated quadratic form, i.e., $Q_K(u) := \int_{I^2} K(s_1, s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$. The following assertions are equivalent:*

1. *There exists $c > 0$ such that for every $u \in L^2(I)$ with $\int_a^b u(t)\,\mathrm{d}t = 0$, $Q_K(u) > c\|U\|^2_{L^2(I)}$, where $U(t) := \int_a^t u(s)\,\mathrm{d}s$;*
2. $\displaystyle\int_I \phi'(s)\,\mathrm{d}s \int_I \frac{\mathrm{d}s}{\phi'(s)} < (b - a + 2\epsilon^{-1})^2.$

On the other hand, if $\int_I \phi'(s)\,\mathrm{d}s \int_I (\phi'(s))^{-1}\,\mathrm{d}s > (b - a + 2\epsilon^{-1})^2$, there exists $u_1, u_2 \in L^2(I)$ with $\int_I u_1(s)\,\mathrm{d}s = \int_I u_2(s)\,\mathrm{d}s = 0$ such that $Q_K(u_1) > 0$ and $Q_K(u_2) < 0$.

If $\epsilon = 0$, the term $(b - a + 2\epsilon^{-1})^2$ should be understood as $+\infty$. The hypothesis that ϕ' is not constant is useful to avoid some degeneracy several times in the proof, but the result still holds if ϕ' is constant by perturbing ϕ.

We first start by recasting the quadratic form in a more manageable way for us. This is done thanks to the following lemma.

Lemma 30. *Define Q_K as in proposition 29. Then, for every $u \in L^2(I)$ with $\int_0^T u(t)\,\mathrm{d}t = 0$,*

$$Q_K(u) = 2\int_I \phi'(s)(U(s))^2\,\mathrm{d}s + 2\epsilon \int_I \phi'(s)U(s)\,\mathrm{d}s \int_I U(s)\,\mathrm{d}s,$$

where $U(t) := \int_a^t u(s)\,\mathrm{d}s$. We will denote the right-hand side of the expression as $\widetilde{Q}_K(U)$ which makes sense for each $U \in L^2(I)$. With this notation, $Q_K(u) = \widetilde{Q}_K(U)$.

This formula actually holds without the assumption $\phi'(s) \geq c > 0$, with the same proof. Moreover, we see that if $\epsilon = 0$, and $\phi' \geq c > 0$, $Q_K(u) \geq 2c\|U\|^2_{L^2}$, so proposition 29 is trivial in this case. From now on, we assume that $\epsilon \neq 0$.

Sketch of the proof. With K as in proposition 29 and w, R as in lemma 20, routine computations show that $w(s) = 2\phi'(s)$ and $R(s_1, s_2) = \epsilon(\phi'(s_1) + \phi'(s_2))$. The terms $g(s)$ and $K(b, b)$ do not matter since $U(b) = 0$. □

The expression of this corollary suggests that we work in the weighted space $L^2_{\phi'} := L^2(I, \phi'(s)\,\mathrm{d}s)$. This is where the hypothesis $\phi'(s) > 0$ is useful: to make sense of this space. We will denote $\|\cdot\|_{\phi'}$ the norm in $L^2_{\phi'}$ and $(\cdot,\cdot)_{\phi'}$ the scalar product. The main consequence of working in this space is that on a space of codimension 2, $Q_K(u) = 2\|U\|^2_{\phi'}$.

Lemma 31. *Let $\widetilde{Q}_K$ as in lemma 30. Let S be the symmetric operator (for the $L^2_{\phi'}$ scalar product) associated with $\widetilde{Q}_K$. Let $E := \{U \in L^2_{\phi'}, \int_I U(s)\,\mathrm{d}s = \int_I \phi'(s)U(s)\,\mathrm{d}s = 0\}$ and $F := \mathrm{Span}(1, (\phi')^{-1})$. Then:*

- *E is the orthogonal of F (for the $L^2_{\phi'}$ scalar product);*
- *E and F are stable by S;*
- *the restriction of S on E is $S_{|E} = 2I$.*

Sketch of the proof. The orthogonality of E and F results from simple computations. Since E is of codimension 2, $E + F = L^2_{\phi'}$. For the other two points, let us denote $M(U)$ the constant function equal to $\fint_I U(s)\,\mathrm{d}s$ and M^* the adjoint of this operator M for the $L^2_{\phi'}$-scalar product. Routine computations show that

$$S(U) = 2U + \epsilon(M(U) + M^*(U)) = 2U + \epsilon\left(\fint_I U(s)\,\mathrm{d}s + \frac{1}{\phi'}\fint_I \phi'(s)U(s)\,\mathrm{d}s\right).$$

With this expression of S, the last two points are immediate. □

With these lemmas, we can prove proposition 29.

Sketch of the proof of proposition 29. The main idea is that according to lemma 31, the only possible counter examples to the coercivity inequality $Q_K(U) \geq c\|U\|^2_{\phi'}$ are in F, thus we are left to study whether a 2×2 matrix is positive.

Let us first compute the matrix of the restriction of $\widetilde{Q}_K$ to F in the basis $(1, (\phi')^{-1})$. Here, we use the fact that ϕ' is not constant; otherwise, the family $(1, (\phi')^{-1})$ would not be linearly independent. For simplicity, write $U_1 := 1$, $U_2 := (\phi')^{-1}$, and $M(U)$ the constant function equal to $\fint_I U(s)\,\mathrm{d}s$. Then,

$$\begin{aligned} A &:= \mathrm{Matrix}_{(U_1,U_2)}(\widetilde{Q}_K)_{|F} \\ &= \begin{pmatrix} 2|U_1|^2 + 2\epsilon(M(U_1), U_1) & 2(U_1, U_2) + \epsilon(M(U_1), U_2) + \epsilon(U_1, M(U_2)) \\ 2(U_1, U_2) + \epsilon(M(U_1), U_2) + \epsilon(U_1, M(U_2)) & 2|U_2|^2 + 2\epsilon(M(U_2), U_2) \end{pmatrix}, \end{aligned}$$

where all the norms and scalar products are taken in $L^2_{\phi'}$. Finally, if we set $\alpha := \fint_I \phi'(s)\,\mathrm{d}s$ and $\beta := \fint_I (\phi'(s))^{-1}\,\mathrm{d}s$, some routine (again) computations prove that this matrix is

$$A = \begin{pmatrix} 2\alpha(1 + \epsilon(b-a)) & 2(b-a) + \epsilon(b-a)^2 + \epsilon\alpha\beta \\ 2(b-a) + \epsilon(b-a)^2 + \epsilon\alpha\beta & 2\beta(1 + \epsilon(b-a)) \end{pmatrix}.$$

To study the positivity of $\widetilde{Q}_K$, we compute the trace and determinant of A. Routine computations show that:

$$\mathrm{Tr}(A) = 2(\alpha + \beta)(1 + \epsilon(b-a)) \tag{47}$$

$$\det(A) = -\epsilon^2(\alpha\beta - (b-a)^2)(\alpha\beta - (b - a - 2\epsilon^{-1})^2) \tag{48}$$

Finally, let us note that thanks to Cauchy-Schwarz inequality, $(b-a)^2 < \alpha\beta$, where the inequality is strict because we assumed that ϕ' is not constant.

Step 1: 1. $\implies$ 2. If assertion 1 holds, $\widetilde{Q}_K$ is positive definite, thus, the matrix A is positive definite. Hence, $\det(A) > 0$. Since $(b-a)^2 < \alpha\beta$, according to the expression (48) of $\det(A)$, we have $\alpha\beta < (b - a + 2\epsilon^{-1})$, which is exactly assertion 2.

Step 2: 2. $\implies$ 1. If assertion 2 holds, according to expression (48) of $\det(A)$ and the fact $(b-a)^2 < \alpha\beta$, we have $\det(A) > 0$. Moreover, since $(b-a)^2 < \alpha\beta < (b - a + 2\epsilon^{-1})^2$, we have $b - a < |b - a + 2\epsilon^{-1}|$, i.e., $b - a < b - a + 2\epsilon^{-1}$ or $b - a < -(b-a) - 2\epsilon^{-1}$. In both cases, we get $1 + \epsilon(b-a) > 0$. Hence, according to the expression (47) of $\mathrm{Tr}(A)$, we have $\mathrm{Tr}(A) > 0$. Thus, A is positive definite. Finally, according to lemma 31, we deduce that for each $U \in L^2$, $\widetilde{Q}_K(U) > c\|U\|^2_{\phi'}$. Since $\phi' \geq c > 0$, the $L^2_{\phi'}$ and L^2 norm are equivalent, hence assertion 1 holds.

Step 3: Last assertion. If $\alpha\beta > (b - a + 2\epsilon^{-1})$, according to expression (48) of $\det(A)$ and the fact $(b-a)^2 < \alpha\beta$, we have $\det(A) < 0$, hence A has a positive and a negative eigenvalue, and so do $\widetilde{Q}_K$. Hence, we can find $\widetilde{U}_1, \widetilde{U}_2 \in L^2(I)$ such that $\widetilde{Q}_K(\widetilde{U}_1) > 0$ and $\widetilde{Q}_K(\widetilde{U}_2) < 0$. By approximating in L^2-norm $\widetilde{U}_i$ by some $U_i \in H^1_0(I)$, we find $U_1, U_2 \in H^1_0(I)$ such that $\widetilde{Q}_K(U_1) > 0$ and $\widetilde{Q}_K(U_2) < 0$. Since $Q_K(U') = \widetilde{Q}_K(U)$, this proves the proposition. □

Acknowledgments. A. Koenig thanks Karine Beauchard, Frédéric Marbach and Mégane Bournissou for many interesting discussions and suggestions to strengthen the results.

A. Koenig is partially supported by a public grant overseen by the French National Research Agency (ANR) as part of the “Investissements d'Avenir”'s program of the Idex PSL reference ANR-10-IDEX-0001-02 PSL.

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